INFOSTUDY'S

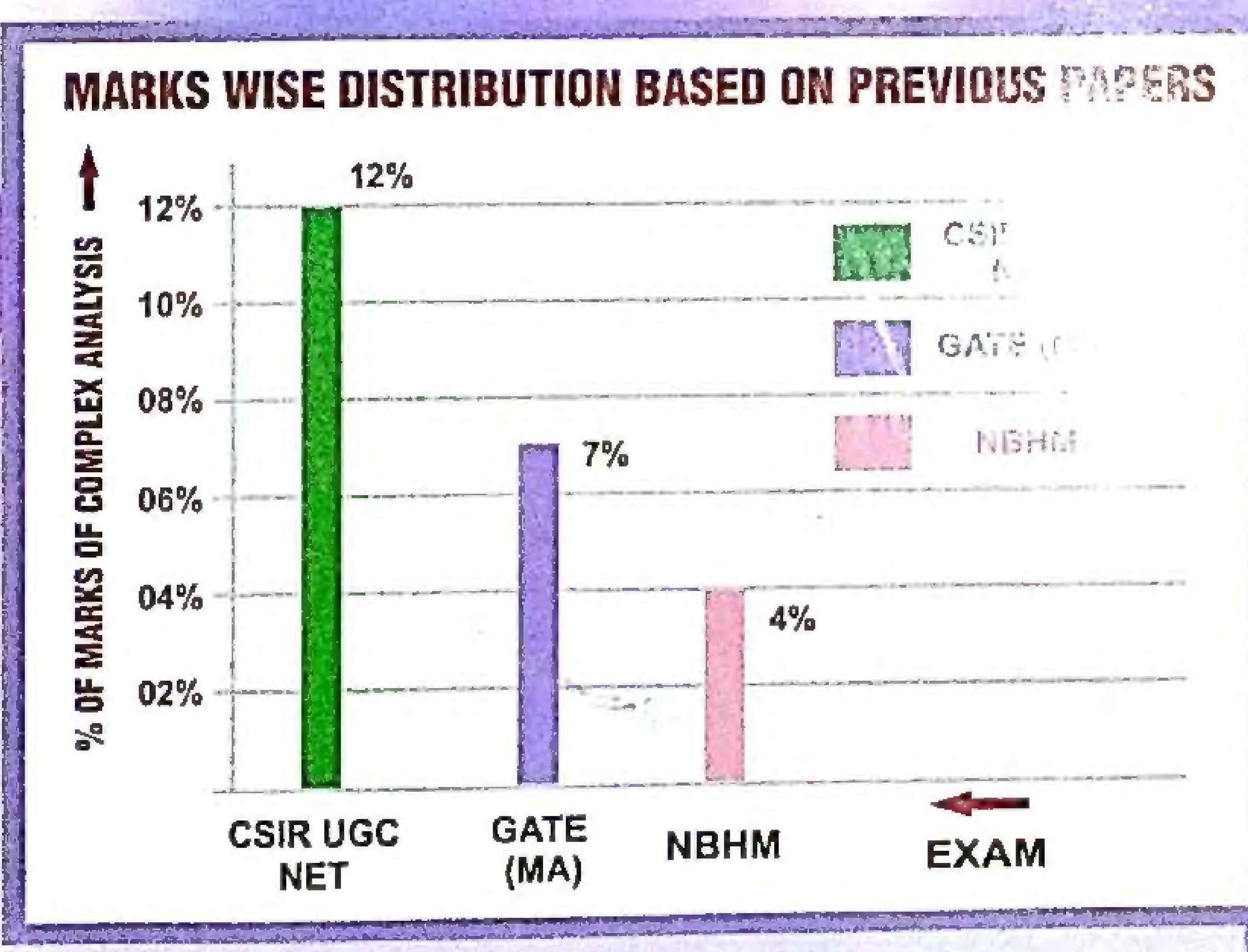
COMPLEX ANALYSIS

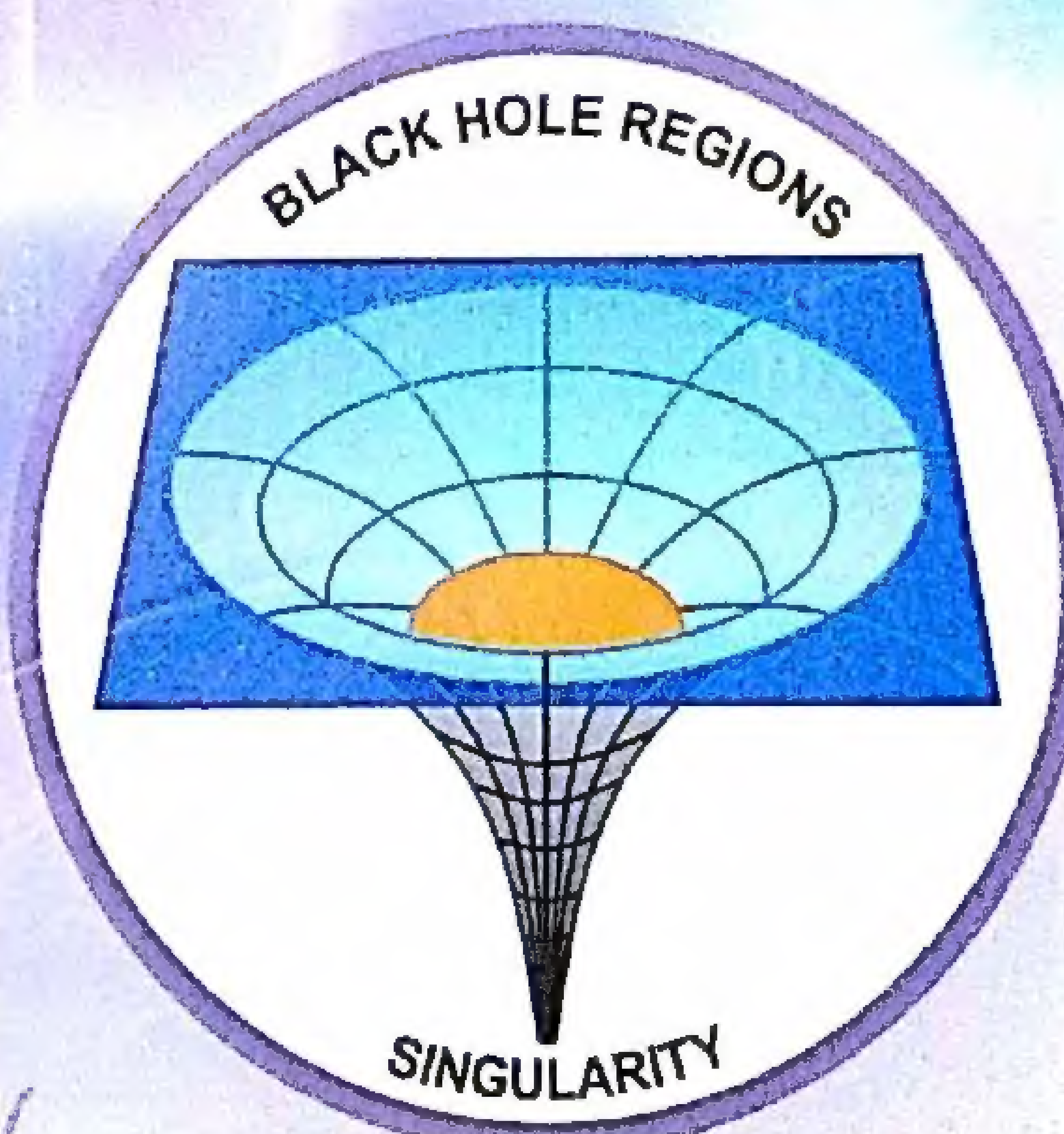
Useful For :
CSIR UGC NET, GATE, IIT-JAM, NBHM, TIFR & Other Exams with Similar Syllabus

By Dr A.P. SINGH

Ph.D. (Mathematics), Indian Young Scientist Awardee, M.Sc. (Gold Medalist), B.Sc. (Gold Medalist)

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint \frac{f(z)dz}{(z-a)^{n+1}} \qquad f(z) = \sin(z)$$





$$\oint_{|z|=1} z^2 dz = 0$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad f(z) = \frac{1}{z}$$

First Edition

INFOSTUDY PUBLICATIONS
BE INFORMED BE LEARNED

COMPLEX ANALYSIS

USEFUL FOR

CSIR UGC NET, GATE, IIT-JAM, NBHM, TIFR & other exams with similar syllabus

First Edition

By

Dr A.P. SINGH

Ph.D. (Mathematics), Indian Young Scientist Awardee
M.Sc. (Gold Medalist) P.U., B.Sc. (Gold Medalist) P.U.

IS INFOSTUDY PUBLICATIONS
BE INFORMED BE LEARNED

Phone : +91-9876788051
Email : infostudypublications@gmail.com

Price : ₹ 675
ISBN : 978-81-933363-5-9

Printed and bound in India by
BANSAL GRAPHICS
Chandigarh

For further information on
INFOSTUDY PUBLICATIONS
Visit - www.infostudypublications.com

Preface

This book is designed for the students who are preparing for various National Level competitive examinations and also inspires to enter into Ph.D. Programs by qualifying the various entrance exams.

This book starts with basic fundamental definitions of complex numbers and then the concepts of elementary functions, complex integration, power series, convergence and uniform convergence, singularities, residues, contour integration, conformal mappings, bilinear transformations etc. are discussed. The concepts of limit, continuity and differentiablity of complex functions are also explained in detail with good examples. This book aims at helping students to get into an insight view of the complex functions of complex variables and will also enable them to solve the problems by an easy and effective approach.

The practice sets are introduced at the end of the topics which includes a variety of questions from CSIR UGC NET, IIT JAM, TIFR, NBHM and GATE previous year question papers. These questions are carefully selected so that the students can apply mathematical knowledge in solving the questions. In addition to it, the solved examples are also given at the end of every chapter which will help in deep understanding of the topics discussed. The key points provides the quick revision of every chapter. Also, a well-thought question bank, in the form of various assignments is given at the end of each chapter which covers entire prescribed topics, so as to facilitate students to do more and more practice and hence secure good results.

While compiling this book, more stress is given on problem solving technique rather than language or exact mathematical symbols. Any suggestions for the improvement of the book will be highly appreciated.

Dr A.P. Singh

Contents

CHAPTER – 0
[BASIC CONCEPTS OF COMPLEX NUMBERS]

INTRODUCTION

"Necessity is the mother of invention." The inability to fully solve the cubic equations leaded "Gerolamo Cardano" to the discovery of complex numbers. Complex numbers $\mathbb{C}$ are field extensions of real $\mathbb{R}$. In this chapter, we will learn the basic concepts of complex numbers and their algebraic and geometric properties. In polar and exponential form, the modulus and arguments have special importance. We will also learn the equations of standard Loci in complex and the way to find n^{th} roots of any complex number using De Moivre's theorem. In the end, we will discuss the one-one correspondence between the complex plane and the sphere of unit radius.

0.1. DEFINITION OF COMPLEX NUMBER

A number of the form $a+ib$, where a, b are real numbers and $i=\sqrt{-1}$, is called a complex number. A complex number can also be defined as an ordered pair of real numbers a and b and hence can be written as (a, b), where the first number denotes the real part and the second number denotes the imaginary part. If $z=a+ib$, then the real part of z is denoted by Re(z) and the imaginary part of z is denoted by Im(z). For example, $5+3i, -1+i, 0+4i, 4+0i$ etc., are complex numbers. A complex number, $z=a+ib$ is said to be purely real if Im(z) = 0 and purely imaginary if Re(z) = 0. The complex number $0=0+i0$ is both purely real and purely imaginary.

Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal, i.e., $a+ib=c+id$ implies a=c and b=d. However, there is no other relation between complex numbers and hence, the expressions of the type $a+ib \nless$ (or >) $c+id$ are meaningless.

(i) *Euler was the first mathematician to introduce the symbol i (iota) for the square root of –1 with property $i^2=-1$. He also called this symbol as the imaginary unit.*

(ii) *For any positive real number a, we have $\sqrt{-a}=\sqrt{-1\times a}=\sqrt{-1}\sqrt{a}=i\sqrt{a}$*

(iii) *The property $\sqrt{a}\sqrt{b}=\sqrt{ab}$ is valid only if at least one of a and b is non-negative. If a and b are both negative, then $\sqrt{a}\sqrt{b}=-\sqrt{|a|.|b|}$.*

(iv) *Every real number is a complex number.*

(v) *0=0+i0 is both purely real and purely imaginary number.*

(vi) ***Integral powers of iota (i):*** *Since $i=\sqrt{-1}$, hence, we have $i^2=-1, i^3=-i$ and $i^4=1$. To find the value of i^n $(n>4)$, first divide n by 4. Let q be the quotient and r be the remainder,*
i.e., $n=4q+r$, where $0\le r\le 3$
$i^n=i^{4q+r}=(i^4)^q.(i)^r=(1)^q.(i)^r=i^r$
In general, we have $i^{4n}=1, i^{4n+1}=i, i^{4n+2}=-1, i^{4n+3}=-i$, where n is any integer.

0.2. ALGEBRA OF COMPLEX NUMBERS

The sum, difference, product and quotient of any two complex numbers is a complex number, i.e., for any two given complex numbers z_1 and z_2, z_1+z_2, z_1-z_2, $z_1.z_2$, $\frac{z_1}{z_2}(z_2 \neq 0)$ are complex numbers.

Algebraic operations with complex numbers:

Let $z_1 = a+ib$ and $z_2 = c+id$ be two complex numbers.

Addition $(z_1 + z_2)$: $(a+ib)+(c+id)=(a+c)+i(b+d)$

Subtraction $(z_1 - z_2)$: $(a+ib)-(c+id)=(a-c)+i(b-d)$

Multiplication $(z_1 . z_2)$: $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

Division $\left(\frac{z_1}{z_2}\right)$: $\frac{a+ib}{c+id}$ *(where at least one of c and d is non-zero)*

$$\frac{a+ib}{c+id}=\frac{(a+ib)}{(c+id)}\cdot\frac{(c-id)}{(c-id)}=\frac{(ac+bd)}{c^2+d^2}+\frac{i(bc-ad)}{c^2+d^2}.$$

Properties of algebraic operations on complex numbers :

Let z_1, z_2 and z_3 be any three complex numbers, then their algebraic operations satisfy following properties :

(i) *Addition of complex numbers satisfies the commutative and associative properties, i.e., $z_1+z_2=z_2+z_1$ and $(z_1+z_2)+z_3=z_1+(z_2+z_3)$.*

(ii) *Multiplication of complex numbers satisfies the commutative and associative properties, i.e., $z_1z_2=z_2z_1$ and $(z_1z_2)z_3=z_1(z_2z_3)$.*

(iii) *Multiplication of complex numbers is distributive over addition, i.e., $z_1(z_2+z_3)=z_1z_2+z_1z_3$ and $(z_2+z_3)z_1=z_2z_1+z_3z_1$.*

(iv) $z+0=z$ *(0 is additive identity)*

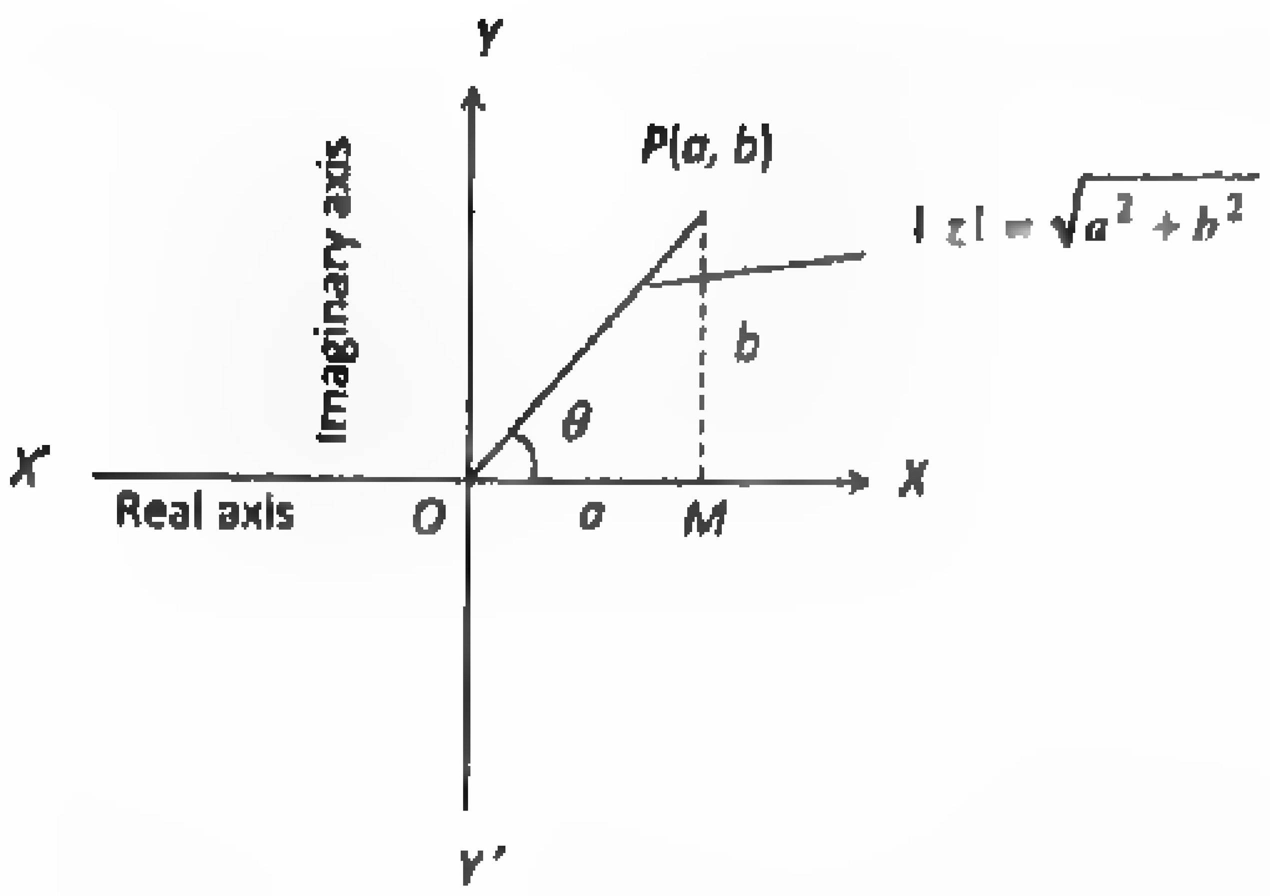
(v) $z+(-z)=0$ *(-z is additive inverse)*

(vi) $z.1=z$ *(1 is multiplicative identity)*

(vii) $z\cdot\frac{1}{z}=1$ *($\frac{1}{z}$ is multiplicative inverse) ; $z \neq 0$*

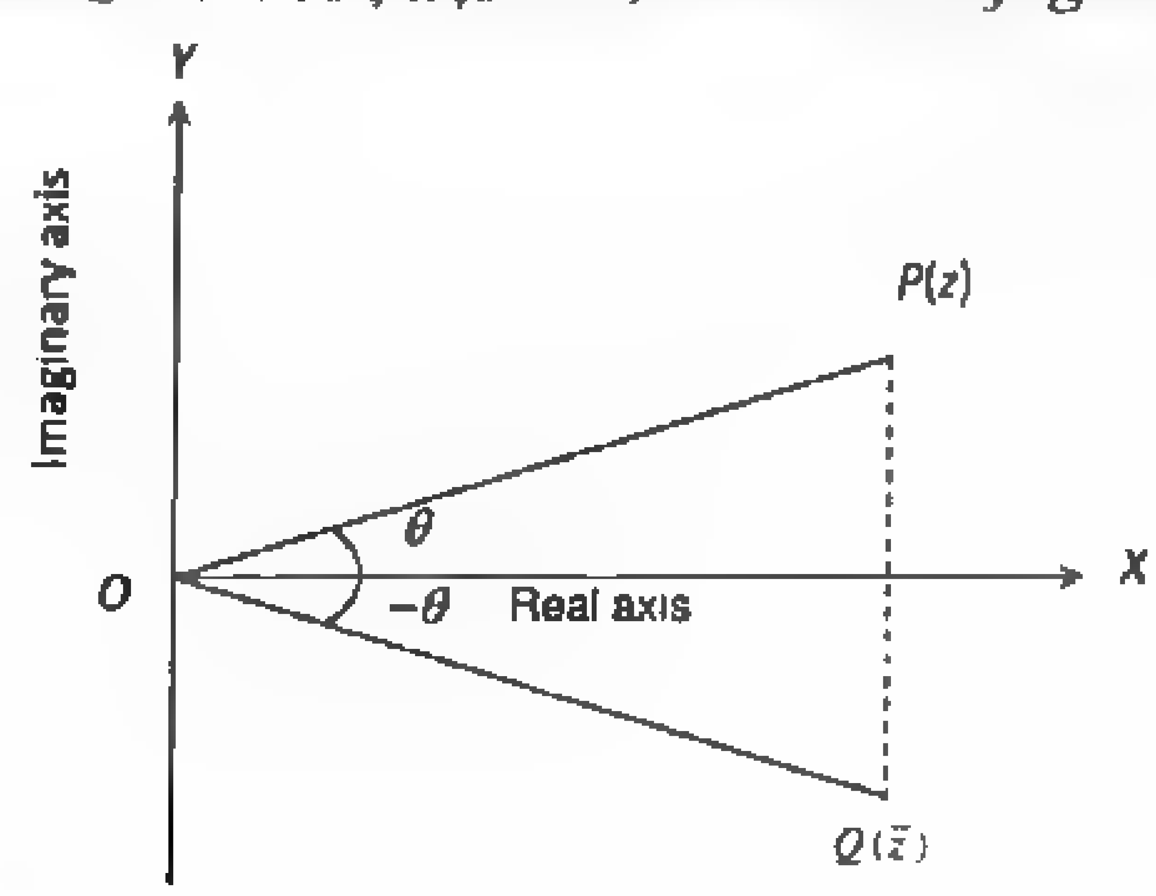
0.3. COMPLEX PLANE

The complex number $z=a+ib=(a,b)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis respectively. This plane is called Argand plane or Argand diagram or complex plane or Gaussian plane.



0.4. CONJUGATE OF A COMPLEX NUMBER

Definition: *If there exists a complex number $z = a + ib$; $a, b \in \mathbb{R}$, then its conjugate is defined as $\bar{z} = a - ib$.*



Hence, we have $\mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$ *and* $\mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$.

Geometrically, the conjugate of z is the reflection or point image of z in the real axis.

Properties of conjugate : *If z, z_1 and z_2 are existing complex numbers, then we have the following results:*

(i) $(\bar{\bar{z}}) = z$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(iv) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$. *In general,* $\overline{z_1.z_2.z_3...z_n} = \bar{z}_1.\bar{z}_2.\bar{z}_3...\bar{z}_n$

(v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

(vi) $(\bar{z})^n = \overline{(z^n)}$

(vii) $z + \bar{z} = 2\,\mathrm{Re}(z) = 2\,\mathrm{Re}(\bar{z}) =$ *purely real*

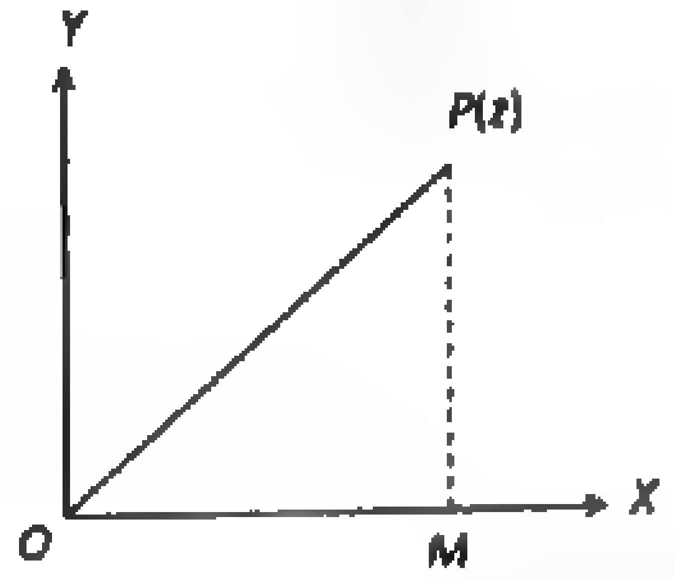
(viii) $z - \bar{z} = 2i\,\mathrm{Im}(z) =$ *purely imaginary*

(ix) $z\bar{z} = |z|^2 = |\bar{z}|^2 =$ *purely real*

(x) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,\mathrm{Re}(z_1\bar{z}_2) = 2\,\mathrm{Re}(\bar{z}_1 z_2)$

0.5. MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = a + ib$ is defined as a positive real number given by $|z| = \sqrt{a^2+b^2}$, where a, b are real numbers. Geometrically, |z| represents the distance of point P from the origin, i.e, |z| = OP.



If $|z|=1$, *the corresponding complex number is known as* **unimodular complex number.** *Clearly, z lies on a circle of unit radius having centre (0, 0).*

Properties of modulus:

(i) $|z|\ge 0 \Rightarrow |z|=0$ if $z=0$ *and* $|z|>0$ if $z\ne 0$

(ii) $-|z|\le \text{Re}(z)\le |z|$ *and* $-|z|\le \text{Im}(z)\le |z|$

(iii) $|z|=|\bar{z}|=|-z|=|-\bar{z}|=|zi|$

(iv) $z\bar{z}=|z|^2=|\bar{z}|^2$

(v) $|z_1z_2|=|z_1||z_2|$. *In general,* $|z_1z_2z_3...z_n|=|z_1||z_2||z_3|...|z_n|$

(vi) $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}, (z_2\ne 0)$

(vii) $|z^n|=|z|^n, n\in\mathbb{N}$

(viii) $|z_1\pm z_2|^2=(z_1\pm z_2)(\bar{z}_1\pm\bar{z}_2)=|z_1|^2+|z_2|^2\pm(z_1\bar{z}_2+\bar{z}_1z_2)=|z_1|^2+|z_2|^2\pm 2\text{Re}(z_1\bar{z}_2)$

(ix) $|z_1+z_2|^2=|z_1|^2+|z_2|^2\Rightarrow\frac{z_1}{z_2}$ *is purely imaginary or* $\text{Re}\left(\frac{z_1}{z_2}\right)=0$

(x) $|z_1+z_2|^2+|z_1-z_2|^2=2\{|z_1|^2+|z_2|^2\}$ *(Law of parallelogram)*

(xi) $||z_1|-|z_2||\le|z_1\pm z_2|\le|z_1|+|z_2|$

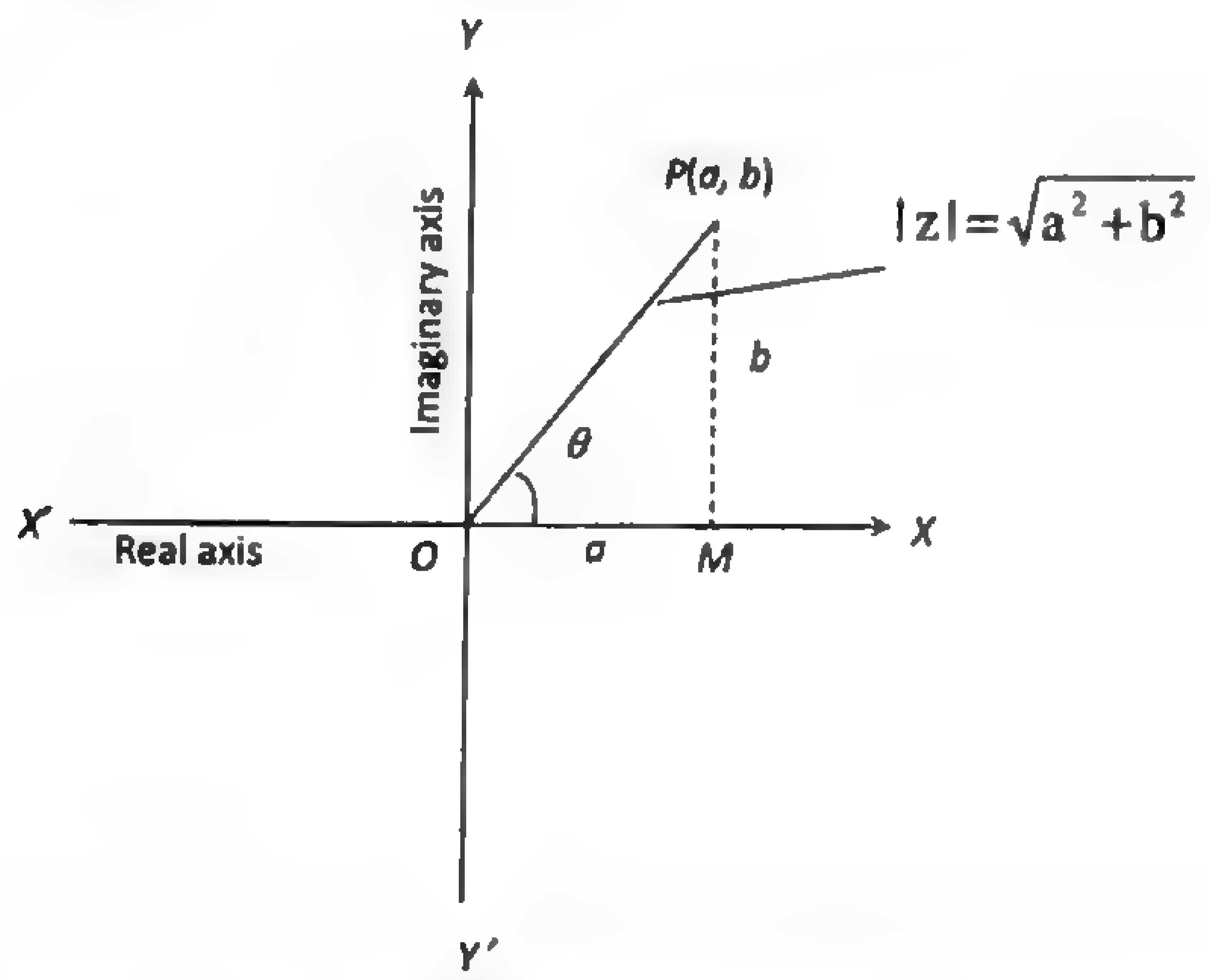
(xii) $|z|=1\Leftrightarrow\bar{z}=\frac{1}{z}$

(xiii) $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

(xiv) $|re^{i\theta}| = r$

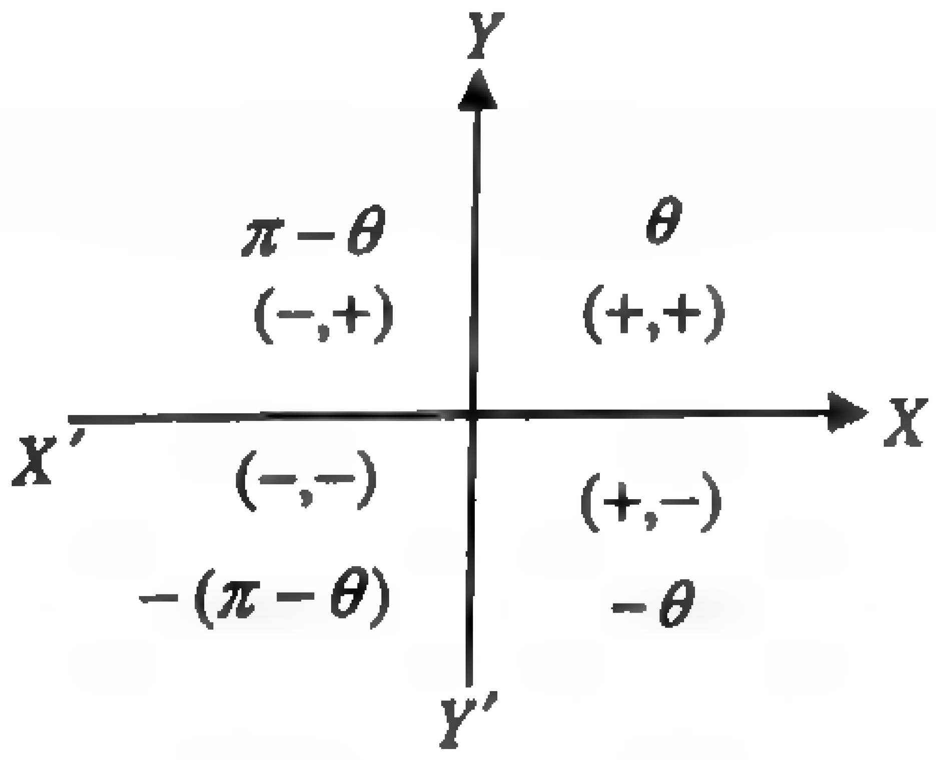
0.6. ARGUMENT OR AMPLITUDE OF A COMPLEX NUMBER

Let $z = a + ib$ be any complex number. If this complex number is represented geometrically by a point P, then the angle made by the line OP with real axis is known as argument or amplitude of z and is expressed as Arg $(z) = \theta = \tan^{-1}\left(\dfrac{b}{a}\right)$, $\theta = \angle POM$



Also, argument of a complex number is not unique, since if θ is a value of the argument, then so is $2n\pi + \theta$, where $n \in \mathbb{Z}$

0.6.1. Principal Value of Arg (z) : *The value θ of the argument, which satisfies the inequality $-\pi < \theta \le \pi$ is called the principal value of argument, where $\theta = \tan^{-1}\left|\dfrac{b}{a}\right|$ (acute angle) and principal values of argument z will be $\theta, \pi - \theta, -\pi + \theta$ and $-\theta$ as the point z lies in the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} quadrants respectively.*



Notation: *General argument of z is denoted by arg(z) and Principal argument of z is denoted by Arg(z).*

Note :

(i) *Principal value of argument of any complex number lies between* $-\pi$ *and* π *i.e.,* $\theta \in (-\pi, \pi]$

(ii) *Argument of a complex number is a many valued function. If* θ *is the argument of a complex number, then* $(2n\pi + \theta)$, $n \in \mathbb{Z}$ *is also argument of complex number. However, the principal argument of complex number is always unique.*

(iii) *Argument of zero is not defined.*

(iv) *If a complex number is multiplied by iota* (i), *then its amplitude will be increased by* $\frac{\pi}{2}$ *and will be decreased by* $\frac{\pi}{2}$, *if it is multiplied by* $(-i)$.

Properties of arguments (amplitudes):

(i) amp (any real positive number) $= 0 + 2n\pi;\ n \in \mathbb{Z}$

(ii) amp (any real negative number) $= \pi + 2n\pi;\ n \in \mathbb{Z}$

(iii) $$\text{amp}(iy) = \begin{cases} \frac{\pi}{2} + 2n\pi; n \in Z, \text{if } y > 0 \\ \frac{-\pi}{2} + 2n\pi; n \in Z, \text{if } y < 0 \end{cases}$$

(iv) $\text{amp}(z) + \text{amp}(\bar{z}) = 0 + 2n\pi;\ n \in \mathbb{Z}$

(v) $\text{amp}(z - \bar{z}) = \pm\frac{\pi}{2} + 2n\pi;\ n \in \mathbb{Z}$

(vi) $\text{amp}(z_1 . z_2) = \text{amp}(z_1) + \text{amp}(z_2) + 2n\pi;\ n \in \mathbb{Z}$

(vii) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2) + 2n\pi;\ z_2 \neq 0,\ n \in \mathbb{Z}$

(viii) $\text{amp}(\bar{z}) = -\text{amp}(z) = \text{amp}\frac{1}{z}$

(ix) $\text{amp}(-z) = \text{amp}(z) \pm \pi$

(x) $\text{amp}(z^n) = n\,\text{amp}(z)$

Example: *Find real θ such that the complex number,* $z = \dfrac{2 + i\cos\theta}{1 - 4i\cos\theta}$ *is*

(i) Purely real *(ii) Purely imaginary*

Solutions: *Given* $z = \dfrac{2 + i\cos\theta}{1 - 4i\cos\theta} = \dfrac{(2 + i\cos\theta)(1 + 4i\cos\theta)}{(1 - 4i\cos\theta)(1 + 4i\cos\theta)} = \dfrac{2-4\cos^2\theta}{1+16\cos^2\theta} + i\dfrac{9\cos\theta}{1+16\cos^2\theta}$

(i) *z is purely real* $\Rightarrow Im(z) = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}.$

(ii) *z is purely imaginary* $\Rightarrow Re(z) = 0$

$\Rightarrow 2 - 4\cos^2\theta = 0 \Rightarrow \cos^2\theta = \dfrac{1}{2} = \cos^2\dfrac{\pi}{4}$

$\Rightarrow \theta = 2n\pi \pm \dfrac{\pi}{4}, n \in \mathbb{Z}$

0.7. *SOME REPRESENTATIONS OF A COMPLEX NUMBER*

A complex number can be represented in the following forms:

(1) ***Geometrical Representation (Cartesian representation):*** *The complex number $z = a + ib = (a,b)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis respectively. We can also say that a complex number z represents a vector because it has magnitude as well as direction.*

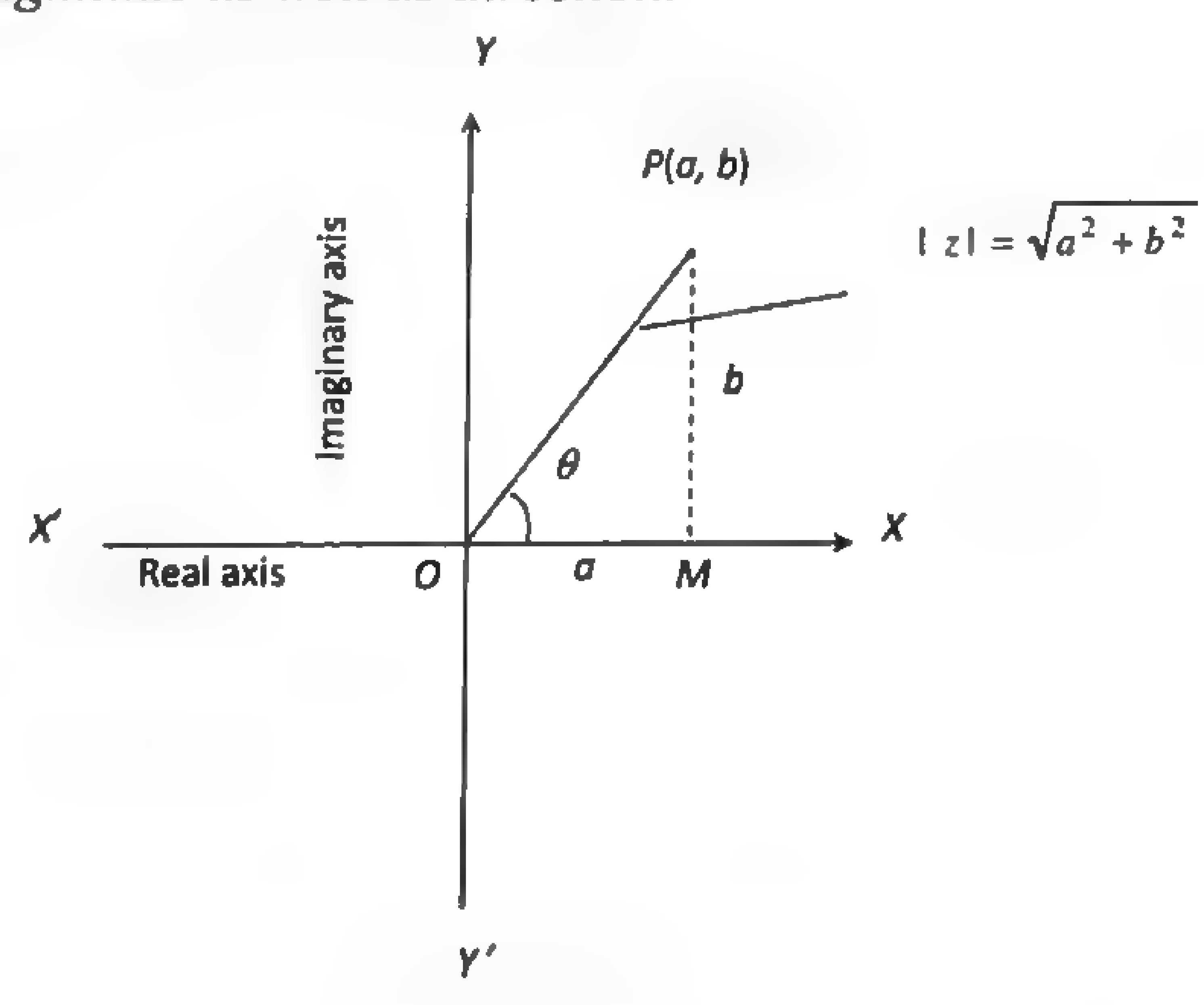


Fig (a)

Distance of any complex number from the origin is called the modulus of complex number and is denoted by |z|, i.e., $|z| = \sqrt{a^2+b^2}$. *Angle of any complex number with positive direction of x–axis is called amplitude or argument of z, i.e., amp* $(z) = arg\ (z) = \tan^{-1}\left(\dfrac{b}{a}\right)$.

(2) **Trigonometrical (Polar) Representation:** *In Fig(a), consider ΔOPM, let $OP = r$, then $a = r\cos\theta$ and $b = r\sin\theta$. Hence, z can be expressed as $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and θ = principal value of argument of z. For general values of the argument $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$.*
Sometimes, $(\cos\theta + i\sin\theta)$ is written in short as $cis\,\theta$.

(3) **Vector representation:** *In Fig (a), if P is the point (a, b) on the argand plane corresponding to the complex number $z = a + ib$. Then $\overrightarrow{OP} = a\hat{i} + b\hat{j}$,*
$\therefore |\vec{OP}| = \sqrt{a^2 + b^2} = |z|$ *and arg (z) = direction of the vector* $\vec{OP} = \tan^{-1}\left(\frac{b}{a}\right)$

(4) **Eulerian representation (Exponential form):** *Since we have $e^{i\theta} = \cos\theta + i\sin\theta$ and thus z can be expressed as $z = re^{i\theta}$, where $|z| = r$ and θ = arg (z).*

0.8. USE OF COMPLEX NUMBERS IN CO-ORDINATE GEOMETRY

(1) **Equation of a Straight Line :**

(i) **Parametric form:** *Equation of a straight line joining the points having affixes z_1 and z_2 is* $z = tz_1 + (1-t)z_2$, where $t \in \mathbb{R}$.

(ii) **Non-parametric form :** *Equation of a straight line joining the points having affixes z_1 and z_2*

is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0.$

or $\dfrac{z - z_1}{z_2 - z_1} = \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

(iii) **General equation of a straight line:** *The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$, where a is complex number and b is real number.*

(2) **Area of a Triangle:** *Area of triangle ABC with vertices A (z_1), B (z_2) and C (z_3) is given by*

$$\Delta = \frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

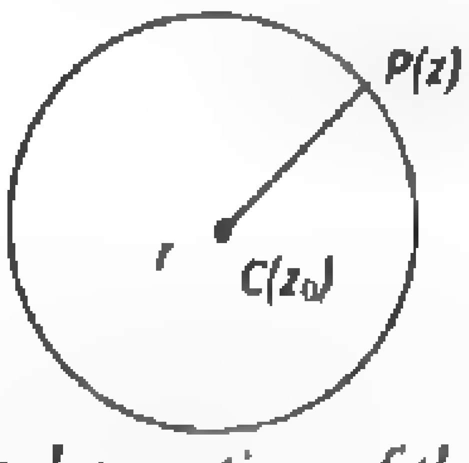
(3) **Condition for Collinearity:** *Three points z_1, z_2 and z_3 will be collinear if there exists a relation $az_1 + bz_2 + cz_3 = 0$ (a, b and c are real), such that $a + b + c = 0$. In other words, three points z_1, z_2 and z_3 are collinear if* $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(4) Equation of a Circle:

(i) The equation of a circle whose centre is at point having affix z_0 and radius r is $|z-z_0|=r$

(ii) If the centre of the circle is at origin and radius r, then its equation is $|z|=r$.

(iii) $|z-z_0|<r$ represents interior of a circle $|z-z_0|=r$ and $|z-z_0|>r$ represents exterior of the circle $|z-z_0|=r$.



(iv) **General Equation of a Circle:** The general equation of the circle is $z\bar{z}+a\bar{z}+\bar{a}z+b=0$, where a is complex number and $b\in\mathbb{R}$.

$\therefore$ Centre and radius are $-a$ and $\sqrt{|a|^2-b}$ respectively.

(v) **Equation of circle in diametric form :** If end points of diameter are represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on the circle then, $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$, which is required equation of circle in diametric form.

0.8.1. Triangle Inequalities:

In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side. By applying this basic concept to the set of complex numbers we are having the following results:

(1) $|z_1+z_2|\le|z_1|+|z_2|$

(2) $|z_1-z_2|\le|z_1|+|z_2|$

(3) $|z_1+z_2|\ge\big||z_1|-|z_2|\big|$

(4) $|z_1-z_2|\ge\big||z_1|-|z_2|\big|$

0.8.2. Geometry of Complex Numbers:

(i) **Distance Formula:** Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ be two complex numbers represented by points P and Q respectively in Argand plane, then

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=|(x_2-x_1)+i(y_2-y_1)|=|z_2-z_1|$$

(ii) If z divides segment joining z_1 and z_2 in the ratio $m_1:m_2$, then $z=\dfrac{m_1z_2+m_2z_1}{m_1+m_2}$ (Internal division)

(iii) $\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)=\theta\Rightarrow\dfrac{z_3-z_1}{z_2-z_1}=\left|\dfrac{z_3-z_1}{z_2-z_1}\right|e^{i\theta}$ (from exponential form of z)

0.8.3. Standard Loci in the Argand plane:

If z is a variable point and z_1,z_2 are two fixed points in the argand plane, then

(i) $|z-z_1|=|z-z_2|\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2

(ii) $|z-z_1|+|z-z_2|=k$, where $k>|z_1-z_2|$, represents an ellipse with foci at z_1 and z_2.

(iii) $|z-z_1|+|z-z_2|=|z_1-z_2|\Rightarrow$ Locus of z is the line segment joining z_1 and z_2.

(iv) $|z-z_1|-|z-z_2|=|z_1-z_2|\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(v) $|z-z_1|-|z-z_2|=k$, where $k<|z_1-z_2|$ represents hyperbola with foci at z_1 and z_2.

(vi) $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2 \Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

0.9. DE MOIVRE'S THEOREM

(1) If n is any rational number, then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

(2) If $z=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)\cdots(\cos\theta_n+i\sin\theta_n)$, then $z=\cos(\theta_1+\theta_2+\theta_3+\dots+\theta_n)+i\sin(\theta_1+\theta_2+\theta_3+\dots+\theta_n)$, where $\theta_1,\theta_2,\theta_3,\dots,\theta_n\in\mathbb{R}$.

(3) If $z=r(\cos\theta+i\sin\theta)$ and n is a positive integer, then

$$z^{1/n}=r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)\right],\text{ where } k=0,1,2,3,\dots,(n-1).$$

0.10. SERIES EXPANSION OF SOME TRIGONOMETRIC FUNCTIONS

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\infty=\sin x \qquad 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\infty=\cos x$$

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\infty=\sinh x \qquad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\infty=\cosh x$$

(Gregory Series)

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\infty=\tan^{-1}x \qquad x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\infty=\tanh^{-1}x=\frac{1}{2}\log\frac{1+x}{1-x}$$

0.11. ROOTS OF A COMPLEX NUMBER

0.11.1. n^{th} Roots of a Complex Number z:

Let $z=r(\cos\theta+i\sin\theta)$ be a complex number. By using De Moivre's theorem, n^{th} roots having n distinct values of such a complex number are given by $z^{1/n}=r^{1/n}\left[\cos\frac{2k\pi+\theta}{n}+i\sin\frac{2k\pi+\theta}{n}\right]$, where $k=0,1,2,\dots,(n-1)$.

Properties of n^{th} roots of z:

(i) All roots of $z^{\frac{1}{n}}$ are in geometrical progression with common ratio $e^{\frac{2\pi i}{n}}$.

(ii) Sum of all roots of $z^{\frac{1}{n}}$ is always equal to zero.

(iii) Product of all roots of $z^{\frac{1}{n}}=(-1)^{n-1}z$.

(iv) Modulus of all roots of $z^{\frac{1}{n}}$ are equal and each equal to $r^{1/n}$ or $|z|^{\frac{1}{n}}$.

(v) Amplitude of all the roots of $z^{\frac{1}{n}}$ are in A.P. with common difference $\frac{2\pi}{n}$.

(vi) All roots of $z^{1/n}$ lies on the circumference of a circle whose centre is origin and radius equal to $|z|^{\frac{1}{n}}$. Also, these roots divides the circle into n equal parts and forms a polygon of n sides.

0.11.2. n^{th} Roots of Unity:

n^{th} roots of unity are given by the solution set of the equation $z^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$

$z = [\cos 2k\pi + i\sin 2k\pi]^{1/n}$

$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, *where* $k = 0,1,2,....,(n-1)$.

Properties of n^{th} roots of unity:

(i) Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i(2\pi/n)}$, n^{th} roots of unity can be expressed in the form of a sequence, i.e., $1, \alpha, \alpha^2, ..., \alpha^{n-1}$. Clearly, the sequence is G.P. with common ratio α, i.e., $e^{i(2\pi/n)}$.

(ii) The sum of all n roots of unity is zero, i.e., $1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0$.

(iii) Product of all n roots of unity is $(-1)^{n-1}$.

(iv) Sum of p^{th} power of n roots of unity

$$1 + \alpha^p + \alpha^{2p} + ... + \alpha^{(n-1)p} = \begin{cases} 0, & \text{when } p \text{ is not multiple of } n \\ n, & \text{when } p \text{ is a multiple of } n \end{cases}$$

The n, n^{th} roots of unity if represented on a complex plane locate their positions at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at origin, one vertex on positive real axis.

0.11.3. Cube Roots of Unity : *Cube roots of unity are the solution set of the equation* $z^3 - 1 = 0$

$\Rightarrow z = (1)^{\frac{1}{3}} \Rightarrow z = (\cos 0 + i\sin 0)^{\frac{1}{3}} \Rightarrow z = \cos\frac{2k\pi}{3} + i\sin\left(\frac{2k\pi}{3}\right)$, *where* $k = 0,1,2$

Therefore, roots are $1, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ *or* $1, e^{2\pi i/3}, e^{4\pi i/3}$.

Properties of cube roots of unity:

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $1 + \omega^n + \omega^{2n} = \begin{cases} 0, & \text{if } n \text{ is not a multiple of } 3 \\ 3, & \text{if } n \text{ is a multiple of } 3 \end{cases}$

(iv) The cube roots of unity, when represented on complex plane, lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

(v) A complex number $a+ib$, for which $|a:b| = 1:\sqrt{3}$ or $\sqrt{3}:1$, can always be expressed in terms of $1, \omega, \omega^2$.

(vi) Cube roots of -1 are $-1, -\omega, -\omega^2$.

0.11.4. *Fourth Roots of Unity:*

The four, fourth roots of unity are given by the solution set of the equation $z^4 - 1 = 0$

$\Rightarrow (z^2-1)(z^2+1)=0 \Rightarrow z = \pm 1, \pm i$

Fourth roots of unity are vertices of a square which lies on coordinate axes.

Example. *Determine the n, n^{th} roots of unity*

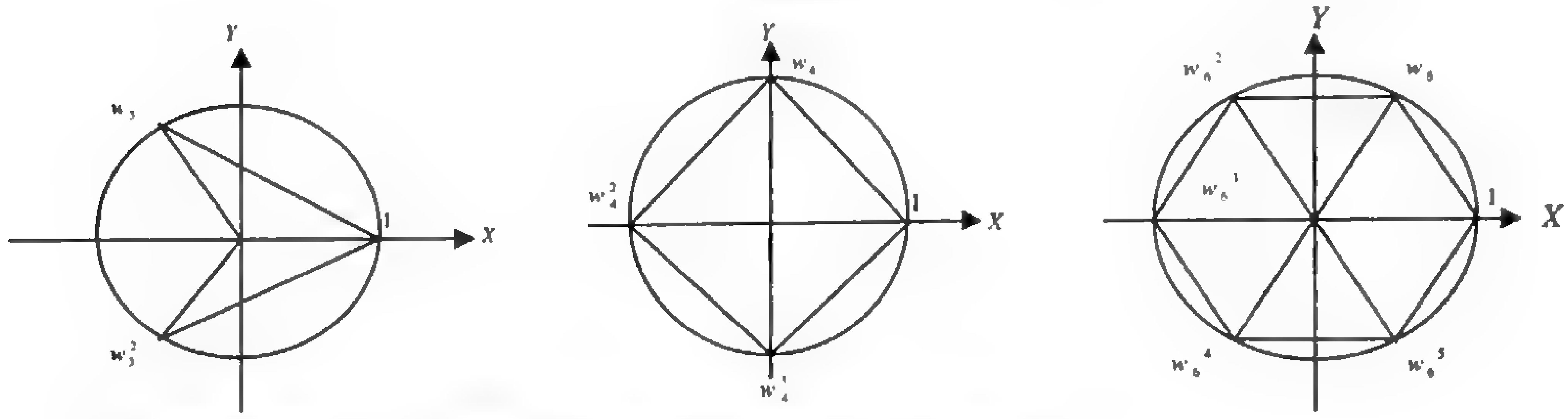
Solution. *$1 = 1 \exp[i(0 + 2k\pi)]$ $(k = 0, \pm 1, \pm 2, \ldots)$*

$$1^{\frac{1}{n}} = \exp\left[i\left(\frac{0}{n}+\frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0,1,2,\ldots,n-1)$$

Thus, the n, nth roots of unity are given by $\exp\left(i\frac{2k\pi}{n}\right)$ $\quad (k=0,1,2,\ldots,n-1)$

These roots are simply, $1, w_n, w_n^2, w_n^3, \ldots, w_n^{n-1}$, *where* $w_n = \exp\left(i\frac{2\pi}{n}\right)$

n = 3 n=4 n=6



The cases when n = 3, 4 and 6 are shown in the figure, where $w_n^{\ n} = 1$

0.11.5. *Powers of Complex Numbers:*

To find the value of any power of a complex number z = x + i y first we express z into the polar form, i.e., $z=x+iy = r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \le \pi$, then we use De Moivre's theorem to find z^n, i.e., $z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$

0.12. *EULER'S FORMULA*

$e^{i\theta} = \cos\theta + i\sin\theta$ *...(1) and* $e^{-i\theta} = \cos\theta - i\sin\theta$ *...(2)*

From (1) and (2), $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ *and* $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Thus, for an integer n, $(e^{i\theta})^n = e^{i(n\theta)} = \cos n\theta + i\sin n\theta$ *and* $(e^{i\theta})^{-n} = e^{i(-n\theta)} = \cos n\theta - i\sin n\theta$

PRACTICE SET

Exercise 1. *Polar form of -3i is*

(a) $\sqrt{3}e^{\frac{\pi i}{2}}$ *(b)* $3e^{\frac{3\pi i}{2}}$ *(c)* $3e^{\frac{\pi i}{2}}$ *(d)* $\sqrt{3}e^{\frac{3\pi i}{2}}$

Exercise 2. Re$\left\{\dfrac{1+i\tan\left(\dfrac{\theta}{2}\right)}{1-i\tan\left(\dfrac{\theta}{2}\right)}\right\}$ *is*

(a) $\cos\theta$ (b) $\cos\left(\dfrac{\theta}{2}\right)$ (c) $\sin\theta$ (d) $\sin\left(\dfrac{\theta}{2}\right)$

Exercise 3. *If* $|z| = |z-1|$, *then*

(a) $\text{Re}\, z = 1$ (b) $\text{Re}\, z = \dfrac{1}{2}$ (c) $\text{Im}\, z = 1$ (d) $\text{Im}\, z = \dfrac{1}{2}$

Exercise 4. *The solution of the equation* $|z| - z = 1 + 2i$ *is*

(a) $1-2i$ (b) $2-\dfrac{3}{2}i$ (c) $\dfrac{3}{2}+2i$ (d) $\dfrac{3}{2}-2i$

Exercise 5. *Principal value of arg* $(1+i)^2$ *is* π/2

Exercise 6. $|z-2i| + |z+2i| = 6$ *is a/an*

(a) circle (b) straight line (c) ellipse (d) hyperbola

Exercise 7. *The value of* $(\omega^3+\omega^4+\omega^5)\left(\dfrac{1}{\omega^3}+\dfrac{1}{\omega^4}+\dfrac{1}{\omega^5}\right)$, *where* ω *is cube root of unity, is*

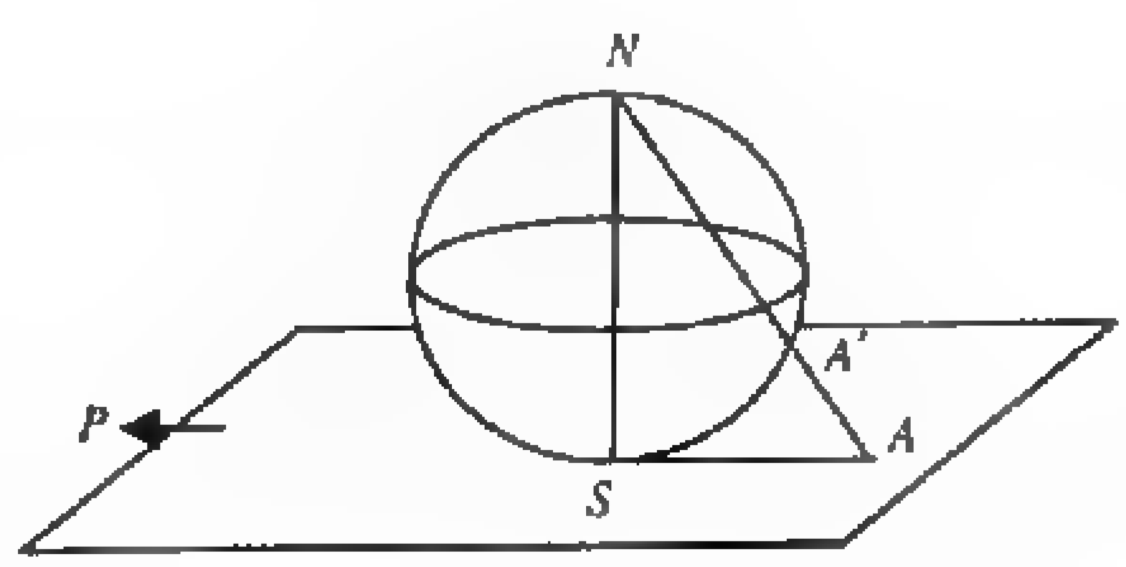
(a) 1 (b) 0 (c) 2 (d) none of these

Exercise 8. *If* $1, \omega_1, \omega_2, \omega_3, \ldots, \omega_{n-1}$ *are the n nth roots of unity, then the value of*

$(1-\omega_1)(1-\omega_2)(1-\omega_3)\ldots(1-\omega_{n-1})$ *is*

(a) n (b) 1 (c) 0 (d) $2n$

0.13. SPHERICAL REPRESENTATION OF COMPLEX NUMBERS (STEREOGRAPHIC PROJECTION)



Let P be the complex plane and consider a unit sphere S (radius one) tangent to plane P at z=0. The diameter NS is perpendicular to P and we call points N and S the north and south poles of S.

Corresponding to any point A on P we can construct line NA intersecting S at point A'. Thus to each point of the complex plane P there corresponds one and only one point of the sphere S and we can

represent any complex number by a point on the sphere. For completeness, we say that the point N itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called the extended complex plane.

The above method for mapping the plane on to the sphere is called stereographic projection. We have the following result.

Important Result: *Under the stereographic projection of points on the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ to the extended complex plane $z = x + iy$, the point $z = x + iy$ corresponds to the point*

$$\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$ *on the sphere. Conversely, any point (X, Y, Z) on the sphere corresponds to the complex number* $z = \frac{X+iY}{1-Z}$

0.14. CHORDAL DISTANCE

Let z_1 and z_2 in the argand plane corresponds stereographically to z_1 and z_2 respectively. Distance between z_1 and z_2, denoted by $\chi(z_1,z_2)$ is referred to as chordal distance between z_1 and z_2.

In other words, the chordal distance between two complex numbers is defined as the distance between corresponding points on sphere. This is always less than or equal to the diameter of the sphere. Chordal distance $\chi(z_1,z_2)$ between z_1 and z_2 is calculated by the following formula

$$\chi(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2} \times \sqrt{1+|z_2|^2}}$$

If $\chi(z_1,z_2) = 1$, *then* $\frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2} \times \sqrt{1+|z_2|^2}} = 1$

$$\Rightarrow |z_1 - z_2| = \sqrt{1+|z_1|^2}\,\sqrt{1+|z_2|^2} \quad \Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (1+|z_1|^2)(1+|z_2|^2)$$

$$\Rightarrow z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2 = 1 + z_2\bar{z}_2 + z_1\bar{z}_1 + z_1 z_2\bar{z}_1\bar{z}_2 \Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 + z_1 z_2\bar{z}_1\bar{z}_2 + 1 = 0$$

$$\Rightarrow z_1\bar{z}_2(z_2\bar{z}_1 + 1) + (z_2\bar{z}_1 + 1) = 0 \Rightarrow (z_1\bar{z}_2 + 1)(z_2\bar{z}_1 + 1) = 0$$

$$\Rightarrow z_1\bar{z}_2 = -1 \Rightarrow z_2\bar{z}_1 = -1$$

Thus, if on the sphere two points z_1 and z_2 are diametrically opposite, then $z_1\bar{z}_2 = -1$

Note: *Chordal distance between z and ∞ is denoted by $\chi(z,\infty)$ and is given by* $\chi(z,\infty) - 1/\ \frac{1}{\sqrt{1+|z|^2}}$

For Example. *Find the chordal distance between $z_1 = 1+i$, $z_2 = \infty$*

Solution: *Chordal distance* $\chi(z,\infty) = \frac{1}{\sqrt{1+|z|^2}} = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$

Antipodal Points:

Two points $z_1, z_2 \in \mathbb{C}$ are said to be antipodal points if their chordal distance is equal to the diameter of the sphere.

- *Two points z_1, z_2 are antipodal iff* $z_1\bar{z}_2 = -1$.
- *Two complex numbers z_1 and z_2 are said to be positionally equal, if $|z_1| = |z_2|$ and $\arg(z_1) - \arg(z_2) = 2n\pi$, $n \in \mathbb{Z}$.*

KEY POINTS

If z is a complex number, then

- $z+\bar{z}=2Re(z)$ and $z-\bar{z}=2iIm(z)$
- $z\bar{z}=|z|^2$
- *Triangle inequality* : $||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|$
- $arg\ (z_1z_2)= arg(z_1)+arg(z_2)+2n\pi$
  $arg\left(\frac{z_1}{z_2}\right)=arg\ (z_1)-arg(z_2)+2n\pi$, where $z_2\neq 0, n\in\mathbb{Z}$
- *arg (0) is not defined*
- $|z-z_1|+|z-z_2|=k$, where $k>|z_1-z_2|$ *represents an ellipse with foci at* z_1 *and* z_2
- $|z-z_1|+|z-z_2|=|z_1-z_2|$ *represents the line segment joining* z_1 *and* z_2
- $|z-z_1|-|z-z_2|=k$ *where* $0<k<|z_1-z_2|$ *represents hyperbola with foci at* z_1 *and* z_2
- ***De Moivre's theorem*** *: For any rational number n,* $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$
- *Sum of* n^{th} *roots of unity = 0*
- *Product of* n^{th} *roots of unity =* $(-1)^{n-1}$

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *If* z_1 *and* z_2 *are distinct complex numbers such that* $|z_1|=|z_2|=1$ *and* $z_1+z_2=1$*, then the triangle in the complex plane with* z_1, z_2 *and -1 as vertices* ***(CSIR UGC NET JUNE-2013)***

(a) must be equilateral *(b) must be right-angled*
(c) must be isosceles, but not necessarily equilateral *(d) must be obtuse angled*

Solution: (a) *Take* $z_1=-\omega=\frac{1}{2}-\frac{\sqrt{3}i}{2},\ z_2=-\omega^2=\frac{1}{2}+\frac{\sqrt{3}i}{2},\ z_3=-1$

Here $|z_1|=|z_2|=1$ *and* $z_1+z_2=1$

Evaluate $|z_1-z_2|, |z_2-z_3|$ *and* $|z_3-z_1|$

$$|z_1 - z_2| = \left|\frac{1}{2} - \frac{\sqrt{3}i}{2} - \frac{1}{2} - \frac{\sqrt{3}i}{2}\right| = \sqrt{3}$$

$$|z_2 - z_3| = \left|\frac{1}{2} + \frac{\sqrt{3}i}{2} + 1\right| = \left|\frac{3}{2} + \frac{\sqrt{3}i}{2}\right| = \sqrt{3}$$

$$|z_3 - z_1| = \left|-1 - \frac{1}{2} + \frac{\sqrt{3}i}{2}\right| = \left|\frac{-3}{2} + \frac{\sqrt{3}i}{2}\right| = \sqrt{3}$$

$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

$\Rightarrow$ *length of three sides of a triangle are equal.*

$\Rightarrow$ *The triangle with z_1, z_2 and -1 as vertices is equilateral.*

So options (b), (c), (d) are incorrect

We are left with option (a) which is correct one.

Example 2. *The minimum possible value of $|z|^2+|z-3|^2+|z-6i|^2$, where z is a complex number and $i=\sqrt{-1}$, is*

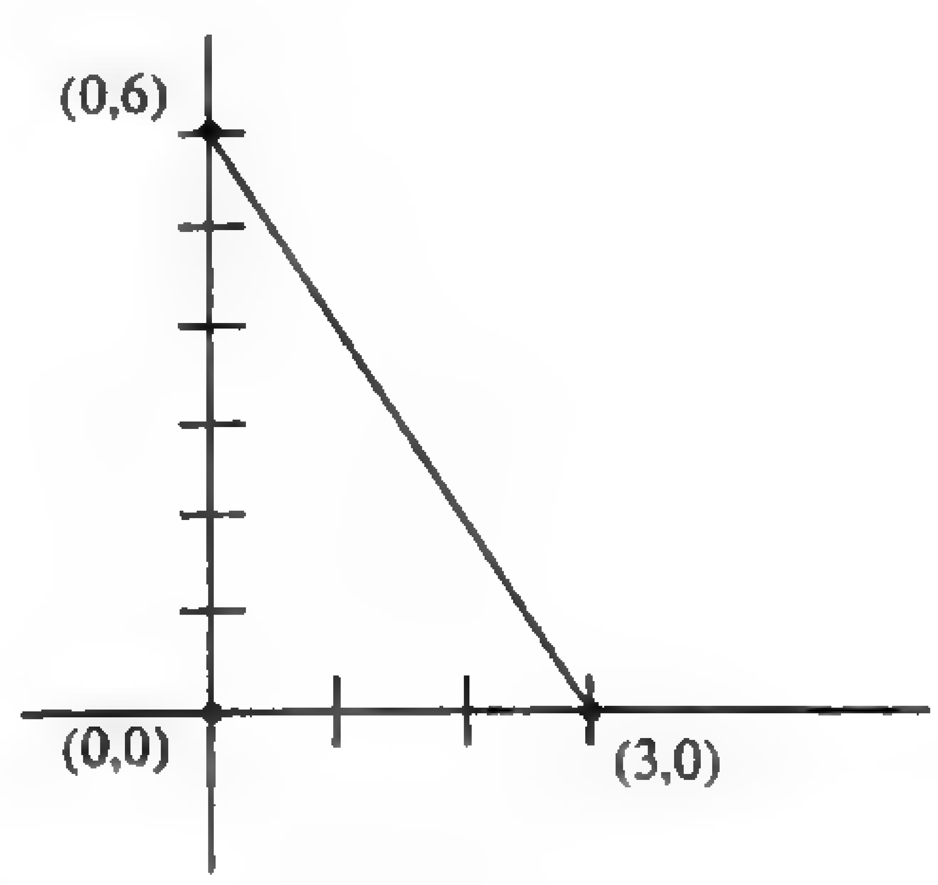
(CSIR UGC NET JUNE-2013)

(a) 15 *(b) 45* *(c) 30* *(d) 20*

Solution: (c) *We will find the minimum possible value of $|z|^2 + |z-3|^2 + |z-6i|^2$*

Let $f(z) = |z|^2 + |z-3|^2 + |z-6i|^2$

For finding minimum value of f(z), we will find centroid of the triangle with vertices (0, 0), (3, 0) and (0, 6)



Centroid $=\left(\frac{0+3+0}{3}, \frac{0+0+6}{3}\right) = (1,2) \Rightarrow z = 1 + 2i$ *will give minimum value.*

Put z = 1 + 2i in f(z), we get

$f(1 + 2i) = (1 + 4) + (4 + 4) + (1 + 16) = 5 + 8 + 17 = 30$

So minimum value of $f(z) = 30$

$\therefore$ *option (c) is correct*

ASSIGNMENT - 0.1

NOTE: CHOOSE THE BEST OPTION

1. The expression $(\cos\theta + i\sin\theta)^n$ is equal to
   (a) $\cos n\theta - i\sin n\theta$
   (b) $\cos n\theta + \sin n\theta$
   (c) $\cos n\theta + i\sin n\theta$
   (d) $\cos n\theta - \sin n\theta$

2. If $\bar{z}$ is the complex conjugate of z, then
   (a) $\bar{z}z = z^2$
   (b) $\bar{z}z = |z|^2$
   (c) $\dfrac{z}{\bar{z}} = |z|$
   (d) none of these

3. For the complex numbers z_1 and z_2, the triangle inequality states
   (a) $|z_1| + |z_2| \geq |z_1 + z_2|$
   (b) $|z_1| + |z_2| \leq |z_1 + z_2|$
   (c) $|z_1 + z_2| = az_1 + bz_2$, where a, b are constants
   (d) none of these

4. For the complex numbers z_1 and z_2, $\arg(z_1/z_2)$ is equal to
   (a) $\arg z_1 + \arg z_2$
   (b) $\arg z_1 - \arg z_2$
   (c) $\arg z_1 / \arg z_2$
   (d) none of these

5. For the complex numbers z_1 and z_2, $\arg z_1 + \arg z_2$ is equal to
   (a) $\arg(z_1 z_2)$
   (b) $\arg(z_1/z_2)$
   (c) $\arg(z_1 - z_2)$
   (d) $\arg(z_1 + z_2)$

6. The reciprocal of $a + ib$ is equal to
   (a) $\dfrac{1}{a^2+b^2} - ib$
   (b) $\dfrac{ib}{a+ib} - a$
   (c) $\dfrac{a^2}{a^2+b^2} - i\dfrac{b^2}{a^2+b^2}$
   (d) $\dfrac{a}{a^2+b^2} - i\dfrac{b}{a^2+b^2}$

7. $\operatorname{Arg}(z) + \operatorname{Arg}(\bar{z})$, when $z \neq 0$ is
   (a) 0
   (b) $-\pi/2$
   (c) $\pi/2$
   (d) πi

8. Value of $\operatorname{Arg}\pi$ is
   (a) real
   (b) 0
   (c) -1
   (d) 1

9. The value of $\operatorname{Re} f(z)$ for $f(z) = 2iz + 6\bar{z}$ is
   (a) $6x + 2y$
   (b) $6x - 2y$
   (c) $6x/2y$
   (d) none of these

10. *The value of* $f(z) = 6\bar{z} + 2iz$ *at* $z = 1/2 + 4i$ *is*

(a) $+(5 + 23i)$ (b) $5 - 23i$
(c) $-(5 + 23i)$ (d) none of these

11. *The correct polar form of the complex number* $1 - i$ *is*

(a) $\sqrt{2}\, e^{\frac{\pi}{4}i}$ (b) $e^{-\frac{\pi}{4}i}$ (c) $\sqrt{2}\, e^{-\frac{\pi}{4}i}$ (d) $e^{\frac{\pi}{4}i}$

12. *The value of* $\left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^4$ *is*

(a) 1 (b) 0 (c) $\cos 4\theta - i\sin 4\theta$ (d) $\cos 8\theta + i\sin 8\theta$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

13. *Which of the following statement is/are true for complex number z?*

(a) $Re\, z = \frac{1}{2}(z + \bar{z})$ (b) $Im\, z = \frac{1}{2i}(z - \bar{z})$
(c) $|\bar{z}| = |z|$ (d) $|z|^2 = z$

14. *Which of the following statement(s) is/are true?*

(a) The multiplicative identity is not unique
(b) Multiplication in $\mathbb{C}$ is associative
(c) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
(d) Sum of two complex numbers is always a complex number

15. *0 is*

(a) purely real (b) purely imaginary
(c) imaginary but not real (d) real but not imaginary

16. *Let* $z = x + iy$*, and* $\bar{z}$ *be its complex conjugate, then for* $y = 0$ *and* $x = 1$

(a) $z = \bar{z}$ (b) $z = \frac{1}{\blacksquare}$
(c) $z = -\bar{z}$ (D) none of these

17. *Let* $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$*, then*

(a) $Re(z_1 + z_2) = Re(z_1) + Re(z_2)$ (b) $Re(z_1 + z_2) = Re(z_1)$
(c) $Re(z_1 + z_2) = Re(z_2)$ (d) $Re(z_1 - z_2) = Re(z_1) - Re(z_2)$

18. *If* $\bar{z}$ *is the conjugate of z, then*

(a) $|\bar{z}| = |z|$ (b) $|\bar{z}| < |z|$
(c) $|\bar{z}| > |z|$ (d) $|\bar{z}|^2 = z\bar{z}$

ASSIGNMENT - 0.2

NOTE: CHOOSE THE BEST OPTION

1. *Complex form of* $\sqrt{3+4i}$ *is*
   (a) $\sqrt{3+i}$ (b) $2-i$
   (c) $2+i$ (d) $\sqrt{3}-i$

2. $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5$, *where* ω *is cube root of unity, is equal to*
   (a) 64 (b) 32
   (c) 16 (d) 8

3. *Principal value of argument of* $(\cos 1200^\circ + i \sin 1200^\circ)$ *is*
   (a) 300° (b) 120°
   (c) -150° (d) 180°

4. *Arg* $(-1+\sqrt{3}\,i)$ *equals*
   (a) $\pi/3$ (b) $\pi/6$
   (c) $2\pi/3$ (d) $5\pi/6$

5. $(1+i)^{10}+(1-i)^{10}=$
   (a) -1 (b) 1
   (c) 0 (d) 2

6. *Let* $\mathbb{C}$ *be the set of complex numbers. Let for any* $z=(x, y)$ *in* $\mathbb{C}$, $z.\bar{z}=z$, *then* $\bar{z}$ *is equal to*
   (a) (0, 1) (b) (1, 0)
   (c) (-1,1) (d) (1, 1)

7. $(\sin\theta + i\cos\theta)^6 =$
   (a) $\sin 6\theta + i\cos 6\theta$ (b) $\cos 6\theta - i\sin 6\theta$
   (c) $-\cos 6\theta + i\sin 6\theta$ (d) $\sin 6\theta - i\cos 6\theta$

8. *If* $\dfrac{4+3i}{3-4i}=x+iy$, *then* $\dfrac{x}{y}$ *is equal to*
   (a) 0 (b) 1
   (c) $\dfrac{4}{3}$ (d) $\dfrac{4}{5}$

9. *The polar representation of* $\left(\dfrac{6+8i}{4-3i}\right)^2$ *is*
   (a) $2\cos(\pi/2 + i\sin\pi/2)$ (b) $4\cos(\pi/2 + i\sin\pi/2)$
   (c) $2(\cos\pi + i\sin\pi)$ (d) $4(\cos\pi + i\sin\pi)$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

10. Which of the following is false for the set of complex number $\mathbb{C}$.

(a) $\alpha + i\beta > 0 + i\beta$, if $\alpha > 0$ and $\beta < 0$

(b) Transitivity law holds in $\mathbb{C}$

(c) Trichotomy law holds in $\mathbb{C}$

(d) $\alpha + i\beta = \gamma + i\delta$, if $\alpha = \gamma$ and $\beta = \delta$

11. For complex number z, $|z+5|^2 + |z-5|^2 = 75$, does not represents

(a) a circle

(b) an ellipse

(c) a triangle

(d) a straight line

12. If z_1 and z_2 are two complex numbers, then $|z_1 + z_2| = |z_1| + |z_2|$ if

(a) $z_1 = z_2$

(b) $z_2 = 0$

(c) $z_1 = \lambda z_2$, for some real number λ, $\lambda > 0$

(d) $z_1 z_2 = 0$ or $z_1 = \lambda z_2$ for some real number λ, $\lambda > 0$

13. Which of the following is true?

(a) $Re(z_1 + z_2) = Re(z_1) + Re(z_2)$

(b) $|z_1 + z_2| \geq |z_1| + |z_2|$

(c) $|z_1 z_2| = |z_1|\,|z_2|$

(d) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

ASSIGNMENT - 0.3

NOTE: CHOOSE THE BEST OPTION

1. If $z = \frac{1+i}{2}$, then the expression $2z^4 - 2z^2 + z + 3$ is equal to

(a) $3 - \frac{i}{2}$
(b) $3 + \frac{i}{2}$
(c) $\frac{3+i}{2}$
(d) $\frac{3-i}{2}$

2. The system of equations $|z + 1 + i| = \sqrt{2}$ and $|z| = 3$ has

(a) no solution
(b) one solution
(c) two solutions
(d) three solutions

3. The points $z_1 = 1 + i$, $z_2 = 1 - i$ and $z_3 = 2 + i$ represents a triangle whose area is

(a) 1
(b) 2
(c) 3
(d) 4

4. The stereographic projection of complex number i on the sphere touching xy-plane and centre at $(0,0,1/2)$ is given by

(a) $\left(\frac{1}{2},\frac{1}{2},0\right)$
(b) $\left(\frac{1}{2},0,\frac{1}{2}\right)$
(c) $\left(0,\frac{1}{2},\frac{1}{2}\right)$
(d) $(0, 0, 1)$

5. By stereographic projection with the south pole at the origin $(0,0,0)$, the point $(1,0,0)$ goes to the complex number

(a) $z = 1$
(b) $z = 1+i$
(c) $z = 1-i$
(d) $z = -1$

6. The value of $|z|$, when $iz^3 + z^2 - z + i = 0$, is

(a) 1
(b) i
(c) 0
(d) 2

7. $|z + i| - |z - i| = k$ represents a hyperbola if

(a) $-2 < k < 2$
(b) $k > 2$
(c) $0 < k < 2$
(d) $2 < k < 4$

8. The region of Argand diagram $|z-1| + |z+1| \le 4$ is bounded by

(a) circle
(b) parabola
(c) ellipse
(d) none of these

9. $|z+4i|+|z-4i|=10$ represents a/an
(a) parabola (b) circle (c) ellipse (d) hyperbola

10. Match the following,
(A) $|z-4i|^2+|z+4i|^2=10$ 1. Ellipse
(B) $|z-4i|^2+|z+4i|^2=10^2$ 2. Straight line
(C) $|z-4i|^2-|z+4i|^2=10$ 3. Circle
(D) $|z-4i|+|z+4i|=10$ 4. None

	A	B	C	D
(a)	4	3	2	1
(b)	4	3	1	2
(c)	1	2	3	4
(d)	1	2	4	3

11. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then which of the following is true?
(a) $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$ (b) $|z_1+z_2|^2+|z_1-z_2|^2\geq 2(|z_1|^2+|z_2|^2)$
(c) $|z_1+z_2|^2+|z_1-z_2|^2\leq 2(|z_1|^2+|z_2|^2)$ (d) $|z_1+z_2|^2-|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

12. The amplitude of the complex number $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ is given by
(a) $\tan^{-1}6$ (b) $\tan^{-1}9$
(c) $\tan^{-1}3$ (d) $\tan^{-1}\frac{3}{2}$

13. If $|z-1|=2$, then the value of $z\bar{z}-z-\bar{z}$ is
(a) 4 (b) 2 (c) 1 (d) 3

14. If $\left|\frac{z-5i}{z+5i}\right|=1$, then $z=x+iy$ lie on
(a) the real axis (b) the straight line $x=5$
(c) the straight line $y=5$ (d) a circle passing through origin

15. If $z=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{35}+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{200}$, then
(a) Re $(z)<0$ (b) Re$(z)=0$
(c) Re $(z)>0$ (d) none of these

16. Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha+\beta>\pi$, then Arg (z_1z_2) is given by
(a) $\alpha+\beta$ (b) $\alpha+\beta-\pi$ (c) $\alpha+\beta+\pi$ (d) $\alpha+\beta-2\pi$

17. If ω is an imaginary cube root of unity, $x=a+b$, $y=a\omega+b\omega^2$ and $z=a\omega^2+b\omega$, then xyz equals to
(a) $a+b$ (b) a^2+b^2 (c) a^4+b^4 (d) a^3+b^3

18. If $x = -2 - \sqrt{3}\, i$, then the value of $2x^4 + 5x^3 + 7x^2 + 41$ is
(A) $4 + \sqrt{3}\, i$
(b) $4 - \sqrt{3}\, i$
(c) $\sqrt{3} + 4i$
(d) $\sqrt{3} - 4i$

19. If $1, \omega, \omega^2$ are the cube roots of unity, then $(x - y)(x - \omega y)(x - \omega^2 y)$ is equal to
(a) $x - y$ (b) $x^2 - y^2$ (c) $x^3 - y^3$ (d) $x^3 + y^3$

20. Common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is/are
(a) ω, ω^2 (b) $1, \omega$ (c) $1, \omega^2$ (d) $1, \omega, \omega^2$

21. The set of points $z \in \mathbb{C}$ for which $|z - 2i| + |z + 2| = 4$ is the conic
(a) hyperbola (b) rectangle (c) square (d) ellipse

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

22. Complex number z such that $z^2 + |z| = 0$
(a) $z = 0$ (b) $z = i$ (c) $z = -i$ (d) $z = 1$

23. If $1, \omega, \omega^2$ are the cube roots of unity, then roots of $(x - 1)^3 + 8 = 0$ are
(a) -1 (b) $1 - 2\omega$ (c) $1 + 2\omega^2$ (d) $1 - 2\omega^2$

24. If $z = \lambda + 3 + i\sqrt{5 - \lambda^2}$, then the locus of z is not given by
(a) a straight line
(b) a circle
(c) an ellipse
(d) a parabola

25. The function $f(z) = z^2\ (0 \le \operatorname{Arg} z \le \pi)$ is not
(a) even
(b) odd
(c) the one about which nothing can be said
(d) both even and odd

ANSWERS TO EXERCISES

(PRACTICE SET)

Exercise 1: (b)	Exercise 2: (a)	Exercise 3: (b)	Exercise 4: (d)
Exercise 5: $\frac{\pi}{2}$	Exercise 6: (c)	Exercise 7: (b)	Exercise 8: (a)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 0.1

1. (c)	2. (b)	3. (a)	4. (b)	5. (a)	6. (d)	7. (a)
8. (b)	9. (b)	10. (c)	11. (c)	12. (d)		
13. (a,b,c)	14. (b,c,d)	15. (a,b)	16. (a,b)	17. (a,d)	18. (a,d)	

ASSIGNMENT - 0.2

1. (c)	2. (b)	3. (b)	4. (c)	5. (c)	6. (b)	7. (c)
8. (a)	9. (d)					
10. (a,b,c)	11. (c,d)	12. (a,b,c,d)	13. (a,c,d)			

ASSIGNMENT - 0.3

1. (a)	2. (a)	3. (a)	4. (c)	5. (a)	6. (a)	7. (c)
8. (c)	9. (c)	10. (a)	11. (a)	12. (b)	13. (d)	14. (a)
15. (a)	16. (d)	17. (d)	18. (b)	19. (c)	20. (a)	21. (d)
22. (a,b,c)	23. (a,b,d)	24. (a,d)	25. (a,b,d)			

CHAPTER - 1
ELEMENTARY FUNCTIONS

INTRODUCTION

In this chapter, we introduce the elementary functions in complex variables which are polynomials, rational functions, trigonometric, hyperbolic, logarithmic functions and complex powers. We will examine the behavior of each of these functions and study their properties. Let us begin with some basic definitions

1.1. SOME BASIC DEFINITIONS

(1) **δ-Neighbourbhood:** *A δ–neighbourhood of a point z_0 in the complex plane is defined as the set of points given by $S=\{z\in\mathbb{C}:|z-z_0|<\delta,\ \delta>0\}$, where δ is called the radius of neighbourhood of z_0 and the set S is denoted by $N_\delta(z_0)$. A deleted δ–neighbourhood of z_0 is a δ–neighbourhood of z_0 in which the point z_0 is excluded. We denote it by $N_\delta(z_0)\setminus z_0=\{z\in\mathbb{C}: 0<|z-z_0|<\delta\}$*

(2) **Limit Point:** *A point z_0 is called limit point of a set S if there exists at least one point of the set S inside the deleted δ–neighbourhood of z, i.e., if every δ–neighbourhood of z_0 contains a point of S other than z_0. Since δ can be any positive number, it follows that S contains infinite number of points.*

It is to be noted that the limit point of the set may or may not belong to the set.

For eg: $z_0 = 0$ is a limit point of $S=\left\{1,\frac{1}{2^2},\frac{1}{3^2},\ldots,\frac{1}{n^2},\ldots\right\}$ but $0=z_0\notin S$.

Also if $S=\{z: |z|<r\}$, then each point on $|z| = r$ is a limit point of S but does not belong to S. Also each $z\in S$ is a limit point of S. The limit point of set S is also called limiting point, cluster point or point of accumulation of S.

(3) **Derived Set:** *The set of all limiting points of a set is called derived set of S and is denoted by S'.*

(4) **Isolated Points:** *Points in a set S which are not limit points are called isolated point of the set S.*

(5) **Closed Set:** *A set S is said to be closed if S contains all its limit points, i.e., if each limit point of S belongs to S. For example, the set of all points z such that $|z|\le r$ is a closed set.*

Note: *Not every point of a closed set need to be a limit point of S.*

For eg., if $S=\{z=0 \text{ or } z=\frac{1}{n^2};\ n\in\mathbb{N}\}$, then $z = 0$ is the only limit point of S and since it belongs to S, So S is closed. However, no other point of S is a limit point of S.

(6) **Bounded Set:** *A set in complex plane is bounded if each point of S lies in $|z| = R$, i.e., if there exists a finite positive number M such that $|z| \leq M$ for every point in S. A set which is not bounded is called an unbounded set.*

(7) **Compact set:** *A set $S \subset \mathbb{C}$ is said to be compact if it is bounded and closed.*

(8) **Interior, Exterior and Boundary Points:**
A point z_0 is called an interior point of a set S, if there exists a δ-neighbourhood of z_0, all of whose points belong to S, i.e., there exists a $\delta > 0$ such that $\{z \in \mathbb{C}: |z-z_0| < \delta\} \subset S$. The set of all interior points of S is denoted by Int S.

A point z_0 is called boundary point of S if each δ-neighbourhood of z_0 contains at least one point belonging to S and at least one point not belonging to S. Thus, if z_0 is a boundary point of S, then each deleted neighbourhood of z_0 intersects S and its complement. The set of all boundary points of S is denoted by ∂S.

A point z_0 is called an exterior point of S if there exists a neighbourhbood of z_0 which contains no point of S. In other words, a point which is neither an interior point nor a boundary point of S, is an exterior point of S.

(9) **Open Set:** *A set S in $\mathbb{C}$ is said to be open if corresponding to each point of S, there exists a neighbourhood which is contained in S. Thus, if set S is open, then S contains all of its interior points. In other words S is open iff each of its point is an interior point. For example, the set of points z such that $|z| < r$ is an open set.*

Note: *The complement of an open set is a closed set and vice-versa.*

(10) **Closure of a Set:** *If to a set S, all its limit points are added, then the new set is called closure of S. The closure of a set is a closed set.*

(11) **Convex Set:** *A set S in $\mathbb{R}^2$ is said to be convex set if for any two points X_1, $X_2 \in S$, the line segment joining X_1 and X_2 is contained in the set S, i.e., $X = \lambda X_1 + (1-\lambda)X_2 \subseteq S$ for $0 \leq \lambda \leq 1$.*

(12) **Connected Set:** *A set S is said to be connected if it can be expressed as a union of any two disjoint non-empty sets S_1, S_2 such that either S_1 contains a limiting point of S_2 or S_2 contains a limiting point of S_1. Hence, an open set S is said to be connected if any two points of the set can be joined by a path consisting of straight line segments (i.e., a polygonal path), all points of which are in S. For example, every interval of real axis is connected.*

(13) **Domain or open Region:** *An open connected set is called an open region or domain.*

(14) **Region:** *A domain together with some, none or all of its limit points is referred to as a region. If all the limit points are added, then the region is closed and if no limit point is added, then the region is open.*

(15) ***Closed Region:*** *The closure of an open region or domain is called closed region.*

1.2. ELEMENTARY FUNCTIONS

1.2.1. ***Complex Function:*** *f(z) is a function of a complex variable z and is denoted by w, i.e.,* $w = f(z) = u + iv$*, where u and v are real functions of x and y and are called real and imaginary parts respectively of f(z).*

1.2.2. ***Polynomial Functions:*** *Polynomial functions are defined by* $w = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = P(z)$*, where* $a_n \neq 0$, $a_{n-1}, a_{n-2}, \ldots, a_0$ *are complex constants and n is a positive integer called the degree of the polynomial P(z).*

Properties of Polynomials:

(i) *The order of a zero of a polynomial equals the order of its first non-vanishing derivative. Suppose, z = a is a zero of order m of polynomial P(z), then* $P(z)=(z-a)^m Q(z)$, $Q(a)\neq 0$. *Differentiating both sides successively m times and putting z = a, we get* $P'(a) = P''(a) = \ldots = P^{m-1}(a) = 0$ *and* $P^m(a) \neq 0$.

(ii) *If all the zeros of a polynomial lie in a half plane, then all the zeros of its derivative also lie in the same half plane* ***(Luca's Theorem).***

(iii) *If S is the set of all zeros of a polynomial, then all the zeros of its derivative lie on convex hull of S* ***(Gauss Theorem).***

(iv) *Let* $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$*, where* $n \geq 1$ *and* $a_n \neq 0$ *so that P(z) is polynomial of degree one or greater, then the equation P(z)=0 has atleast one root* ***(Fundamental theorem of algebra).***

1.2.3. ***Rational Algebraic Functions:*** *Rational algebraic functions are defined by* $w=\dfrac{P(z)}{Q(z)}$*, where P(z) and Q(z) are polynomials without common zeros. w is sometimes called a rational transformation.*
The real and imaginary parts of w are rational functions of x and y. If we take P(z) = az+b and Q(z)=1, then w=az+b is called linear transformation.

The special case $w = \dfrac{az+b}{cz+d}$*, where* $ad - bc \neq 0$ *is often called a bilinear or fractional linear transformation.*

1.2.4. ***Exponential Function:*** *Exponential function is defined by* $w=e^z = e^{x+iy} = e^x(\cos y + i \sin y)$*, where e=2.71828... is the natural base of logarithms. Series expansion of exponential function is given by*

$$\exp(z) \text{ or } e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \infty = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \ldots(1)$$

Properties of Exponential Function:

(a) $e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y) \Rightarrow |e^z| = |e^x| = e^x \quad (\because e^x > 0)$ *and arg* $(e^z) = y$

(b) $e^{\bar{z}} = \overline{(e^z)}$

(c) $e^{z_1} e^{z_2} = e^{z_1+z_2} \quad e^{\frac{z_1}{z_2}} = e^{z_1 - z_2}$

(d) e^z *is periodic function having imaginary period* $2\pi i$, $\because e^{z+2i\pi} = e^x e^{i(y+2\pi)} = e^x e^{iy} = e^z$

(e) *Exponential form of* $z = re^{i\theta}$.

(f) $e^z \neq 0 \ \forall z \in \mathbb{C} \ \because |e^z| = e^x > 0 \ \forall x \in \mathbb{R}$

1.2.5. ***Trigonometric Functions:*** *We define the trigonometric or circular functions sinz, cosz, etc., in terms of exponential functions are defined as follows:*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}$$

$$\operatorname{cosec} z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Using the series expansion of e^z, *we have the following series for sinz and cosz as:*

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + \frac{(-1)^n z^{2n}}{(2n)!} + \ldots$$

Properties of trigonometric functions:

(i) $\sin^2 z + \cos^2 z = 1$

(ii) $1 + \tan^2 z = \sec^2 z$

(iii) $1 + \cot^2 z = \operatorname{cosec}^2 z$

(iv) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\tan(-z) = -\tan z$

(v) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

(vi) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

(vii) $\tan(z_1 \pm z_2) = \dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$

1.2.6. Hyperbolic Functions: *The hyperbolic functions in terms of exponential functions are defined as follows:*

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$$

$$\operatorname{cosech} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

Using the series expansion of e^z, we have the following series for sinhz and coshz as:

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots + \frac{z^{2n+1}}{(2n+1)!} + \dots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} \dots + \frac{z^{2n}}{(2n)!} + \dots$$

Properties of hyperbolic functions:

(i) *sinhz and coshz are periodic functions with imaginary period $2\pi i$*

(ii) *coshz is an even function while sinhz is an odd function*

(iii) $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

(iv) $\cosh^2 z - \sinh^2 z = 1$

(v) $1 - \tanh^2 z - \operatorname{sech}^2 z$

(vi) $\coth^2 z - 1 = \operatorname{cosech}^2 z$

(vii) $\sinh(-z) = -\sinh z$

(viii) $\cosh(-z) = \cosh z$

(ix) $\tanh(-z) = -\tanh z$

(x) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(xii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

1.2.7. *Relations between the trigonometric or circular functions and the hyperbolic functions:*

$\sin iz = i \sinh z$
$\cos iz = \cosh z$
$\tan iz = i \tanh z$
$\sinh iz = i \sin z$
$\cosh iz = \cos z$
$\tanh iz = i \tan z$

1.2.8. *Logarithmic Function:* *If $z = e^w$, then we write $w = \ell n z$, called the natural logarithm of z. Thus the natural logarithmic function is the inverse of the exponential function and can be defined by $w = \ell n z = \ell n r + i(\theta + 2k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$, where $z = re^{i\theta} = re^{i(\theta+2k\pi)}$, $k \in \mathbb{Z}$.*

Note: *The function, $f(z) = \ln z$ is defined in $\mathbb{C}\setminus\{0\}$ and every non-zero complex number has infinitely many logarithms which differ from each other by integral multiples of $2\pi i$. This implies that $w = \ln z$ is not function in general and is infact a multi-valued relation, with infinitely many values for each $z \neq 0$. In order to make $w = \ln z$ single-valued, the term principal value or principal branch of $\ln z$ is used which is defined as $\ln r + i\theta$, where $0 \leq \theta < 2\pi$. However, any interval, say $(-\pi, \pi]$ of length 2π can be used.*
The logarithmic function can be defined for real bases other than e.
Thus if $z = a^w$, then $w = \log_a z$, where $a > 0$ and $a \neq 0, 1$. In this case $z = e^{w\ell n a}$ and so $w = \dfrac{\ell n z}{\ell n a}$.

1.2.9. *Inverse Trigonometric Functions:* *If $z = \sin w$, then $w = \sin^{-1} z$ is called the inverse of sin z or arc of sinz. Similarly, other inverse trigonometric or circular functions $\cos^{-1} z$, $\tan^{-1} z$, etc. are defined. Since the logarithmic function is multivalued, hence these functions too are multivalued. These functions, can be expressed in terms of natural logarithms as follows. In all cases, an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm is omitted.*

$$\sin^{-1} z = \frac{1}{i} \ell n \left(iz + \sqrt{1-z^2}\right)$$

$$\operatorname{cosec}^{-1} z = \frac{1}{i} \ell n \left(\frac{i + \sqrt{z^2-1}}{z}\right)$$

$$\cos^{-1} z = \frac{1}{i} \ell n \left(z + \sqrt{z^2-1}\right)$$

$$\sec^{-1} z = \frac{1}{i} \ell n \left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tan^{-1} z = \frac{1}{2i} \ell n \left(\frac{1+iz}{1-iz}\right)$$

$$\cot^{-1} z = \frac{1}{2i}\,\ell n\left(\frac{z+i}{z-i}\right)$$

1.2.10. Inverse Hyperbolic Functions: *If $z = \sinh w$, then $w = \sinh^{-1} z$ is called the inverse hyperbolic sine of z. Similarly, other inverse hyperbolic functions $\cosh^{-1} z$, $\tanh^{-1} z$, etc. are defined. These functions are also multi-valued and hence can be expressed in terms of natural logarithms as follows. In all cases, an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm is omitted.*

$$\sinh^{-1} z = \ell n\,(z + \sqrt{z^2+1})$$

$$\operatorname{cosec}^{-1} z = \ell n\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

$$\cosh^{-1} z = \ell n\,(z + \sqrt{z^2-1})$$

$$\operatorname{sech}^{-1} z = \ell n\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\,\ell n\left(\frac{1+z}{1-z}\right),$$

$$\coth^{-1} z = \frac{1}{2}\,\ell n\left(\frac{z+1}{z-1}\right)$$

1.2.11. The Function z^α, *where α may be complex, is defined as $z^\alpha = e^{\alpha \ln z}$, since $\ln z = \ln|z| + i\,(\phi + 2n\pi)$ where ϕ denotes the principal value of arg(z).*

$$\therefore\ z^\alpha = e^{\alpha(\ln|z| + i(\phi + 2n\pi))} = e^{(\alpha \ln|z| + i\alpha\phi + \alpha i 2n\pi)} = e^{\alpha \ln|z|}\, e^{i\alpha\phi}\, e^{2in\pi\alpha}.$$

Thus, in general, z^α is infinite valued and different values of z^α are obtained by giving different integral values to n. Similarly, if f(z) and g(z) are two given functions of z, we define $f(z)^{g(z)} = e^{g(z)\ln f(z)}$. In general, such functions are multiple valued.

1.2.12. Algebraic: *If w is a solution of the polynomial equation $P_n(z)w^n + P_{n-1}(z)w^{n-1} + \ldots + P_1(z)w + P_0(z) = 0$, where $P_n \neq 0$, $P_i(z)$ are polynomials in z for all i and n is a positive integer, then $w = f(z)$ is called an algebraic function of z such type of function involves only algebraic operations like addition, substraction, multiplication and division as well as fractional or rational exponents.*

For Example: *$w = z^{1/3}$ is a solution of the equation $w^3 - z = 0$ and so is an algebraic function of z.*

1.2.13. Transcendental Functions: *Any function which cannot be expressed as a solution of an equation is called a transcendental function. The logarithmic, trigonometric and hyperbolic functions and their corresponding inverse are examples of transcendental functions.*

The functions considered above, together with functions derived from them by a finite number of algebraic operations involving addition, subtraction, multiplication, division are called elementary functions.

1.2.14. Monovalent Function: *A function $f : \mathbb{C} \to \mathbb{C}$ is said to be monovalent iff f is one – one.*

For Example. *In which region e^z is monovalent?*

Solution: *We have* $f(z_1)=f(z_2) \Rightarrow e^{z_1}=e^{z_2} \quad \Rightarrow z_1-z_2=2n\pi i,\ n\in\mathbb{Z}$

$\therefore$ *The required region is* $=\{z : |z_1-z_2|=2n\pi, n\in\mathbb{Z}\}$

PRACTICE SET

Exercise 1. *Let* $E=\{z\in\mathbb{C} : e^z=i\}$*. Then E is*

(a) a singleton
(b) a set of 4 elements
(c) an infinite set
(d) an infinite group under addition

Exercise 2. *The set of limit points of* $\left\{ z\in\mathbb{C} :\ z=\frac{i}{n}, n\in\mathbb{Z} \right\}$ *is*

(a) $\{i\}$
(b) $\{0\}$
(c) singleton
(d) infinitely many

Exercise 3. *Principal value of* $\log(1+i)^2$ *is* ______________.

Exercise 4. $\mathrm{Re}\{(1-i)^{1+i}\}$ *is* e ______________

Exercise 5. *Which of the following sets are convex?*

(a) $\{(x,y)\,|\,x^2+y^2\ge 1\}$
(b) $\{(x,y)\,|\,y^2\ge x\}$
(c) $\{(x,y)\,|\,3x^2+4y^2\le 5\}$
(d) $\{(x,y)\,|\,y\ge 2, y\le 4\}$.

Exercise 6. *The points* $z_1=1+3i$, $z_2=1-3i$ *and* $z_3=2-3i$ *represents a triangle whose area is*

(a) 3
(b) 2
(c) 4
(d) None of these

Exercise 7. *The principal value of* $\log\left(i^{\frac{1}{4}}\right)$ *is* **(GATE-2005)**

(a) $i\pi$
(b) $\frac{i\pi}{2}$
(c) $\frac{i\pi}{4}$
(d) $\frac{i\pi}{8}$

KEY POINTS

- *The order of a zero of a polynomial equals the order of its first non-vanishing derivative.*
- ***Fundamental Theorem of Algebra:*** *Every polynomial* $P(z)$ *of degree n has exactly n roots in* $\mathbb{C}$.
- ***Exponential function,*** e^z *is periodic having imaginary period* $2\pi i$.
- $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$, $\cos z=\dfrac{e^{iz}+e^{-iz}}{2}$, $\sinh z=\dfrac{e^z-e^{-z}}{2}$, $\cosh z=\dfrac{e^z+e^{-z}}{2}$.
- *sinhz and coshz are periodic functions with period* 2π.

- $\log z = \ell n|z| + i\arg(z)$

- *Complex powers :* z^{α}, *where* α *is a complex number can be written as* $e^{\alpha \ln z}$

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *Which of the following is the imaginary part of a possible value of* $\ell n(\sqrt{i})$? *(GATE-2011)*

(a) π (b) $\pi/2$
(c) $\pi/4$ (d) $\pi/8$

Solution: (c) $\log\sqrt{i} = \log\left(e^{i\frac{\pi}{2}}\right)^{1/2} = \log e^{i\frac{\pi}{4}} = i\frac{\pi}{4}$

$\therefore \ \operatorname{Im}(\log\sqrt{i}) = \frac{\pi}{4}$

Example 2. *Let p be a polynomial in 1-complex variable. Suppose all zeroes of p are in the upper half plane* $H = \{z \in \mathbb{C} \mid Im(z) > 0\}$. *Then* *(CSIR UGC NET JUNE-2015)*

(a) $\operatorname{Im}\frac{p'(z)}{p(z)} > 0$ *for* $z \in \mathbb{R}$. (b) $\operatorname{Re} i\frac{p'(z)}{p(z)} < 0$ *for* $z \in \mathbb{R}$

(c) $\operatorname{Im}\frac{p'(z)}{p(z)} > 0$ *for* $z \in \mathbb{C}$, *with* $\operatorname{Im} z < 0$ (d) $\operatorname{Im}\frac{p'(z)}{p(z)} > 0$ *for* $z \in \mathbb{C}$ *with* $\operatorname{Im} z > 0$

Solution: **(a,b,c)** *Take* $p(z) = a(z - z_1)(z - z_2) \ldots (z - z_k)$, *with* $z_1, z_2, \ldots, z_k$ *lie in* $H = \{z \in \mathbb{C} \mid Im(z) > 0\}$

$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n}\frac{1}{z - z_k}$ *and for all* $1 \le k \le n$

$\frac{1}{z - z_k} = \frac{\bar{z} - \bar{z}_k}{|z - z_k|^2} = \frac{(x - x_k) - (y - y_k)i}{|z - z_k|^2}$, *with* $z = x + iy,\ z_k = x_k + iy_k$

$\operatorname{Re}\frac{p'(z)}{p(z)} = \sum_{k=1}^{n}\frac{x - x_k}{|z - z_k|^2},\ \operatorname{Im}\frac{p'(z)}{p(z)} = -\sum_{k=1}^{n}\frac{y - y_k}{|z - z_k|^2}$

For option (a)

For $z \in \mathbb{R}$, *put* $y = 0$, $z = x$

$\operatorname{Im}\frac{p'(z)}{p(z)} = -\sum_{k=1}^{n}\frac{-y_k}{|z - z_k|^2} = \sum_{k=1}^{n}\frac{y_k}{|z - z_k|^2} > 0$, *as* $y_k > 0$ *for* $1 \le k \le n$

$\therefore$ *option (a) is correct*

For option (c)

$\operatorname{Im}\frac{p'(z)}{p(z)} = -\sum_{k=1}^{n}\frac{y - y_k}{|z - z_k|^2} > 0$, *as* $y_k > 0$ *for* $1 \le k \le n$ *and* $Imz < 0$

$\therefore$ *option (c) is correct*

For option (d)

$\text{Im}\dfrac{p'(z)}{p(z)} = -\sum_{k=1}^{n}\dfrac{y-y_k}{|z-z_k|^2}$ *is negative or positive, depends on whether* $Imz > Imz_k$ *or* $Imz < Imz_k$

$\therefore$ *option (d) is incorrect*

$$i\frac{p'(z)}{p(z)} = \sum_{k=1}^{n}\frac{(x-x_k)i}{|z-z_k|^2} + \sum_{k=1}^{n}\frac{y-y_k}{|z-z_k|^2}$$

$\text{Re}\, i\dfrac{p'(z)}{p(z)} = \sum_{k=1}^{n}\dfrac{y-y_k}{|z-z_k|^2} = \sum_{k=1}^{n}\dfrac{y_k}{|z-z_k|^2} < 0$, *for* $z \in \mathbb{R}$ *as* $y_k > 0$

$\Rightarrow$ *option (b) is correct.*

Example 3. *Let P(z), Q(z) be two complex non-constant polynomials of degree m, n respectively. The number of roots of P(z)=P(z)Q(z) counted with multiplicity is equal to:* ***(CSIR UGC NET JUNE – 2016)***

(a) min {m,n} *(b) max {m,n}* *(c) m+n* *(d) m-n*

Solution: (c) *Consider* $P(z) = (z-1)^2$ *and* $Q(z)=z^3$.

Thus, P(z) and Q(z) are two complex non-constant polynomials of degree m=2 and n=3 respectively

$\therefore$ *P(z)=P(z)Q(z)* ...(1)

$\Rightarrow (z-1)^2 = (z-1)^2 z^3$

$\Rightarrow (z-1)^2[1-z^3] = 0$

$\therefore$ *The number of roots of equation (1) counted with multiplicity is equal to 5*

$\therefore$ *Options (a), (b), (d) are incorrect*

Hence, option (c) is correct.

Example 4. *Consider the polynomial* $P(z) = \left(\sum_{n=0}^{5}a_n z^n\right)\left(\sum_{n=0}^{9}b_n z^n\right)$, *where* $a_n, b_n \in \mathbb{R}\ \forall\ n,\ a_5 \neq 0,\ b_9 \neq 0$. *Then counting roots with multiplicity we can conclude that P(z) has* ***(CSIR UGC NET DEC-2016)***

(a) at least two real roots. *(b) 14 complex roots*

(c) no real roots *(d) 12 complex roots.*

Solution: (a,b) $P(z) = \left(\sum_{n=0}^{5}a_n z^n\right)\left(\sum_{n=0}^{9}b_n z^n\right)$, $a_n,\ b_n \in \mathbb{R},\ a_5 \neq 0,\ b_9 \neq 0$, *a polynomial of degree 14*

$\Rightarrow$ *In* $\mathbb{C}$, *all roots exist*

$\Rightarrow$ *option (b) is correct*

Now $\sum_{n=0}^{5}a_n z^n$, *has at least one real root and similarly* $\sum_{n=0}^{9}b_n z^n$ *has at least one real root,*

Otherwise a_n *and* b_n *will no longer remain real.*

$\Rightarrow$ *P(z) has atleast 2 real roots*

Thus, options (a), (b) are correct and (c), (d) are incorrect.

ASSIGNMENT - 1.1

NOTE: CHOOSE THE BEST OPTION

1. The points of unit circle $|z| = 1$ forms a/an
   (a) open set (b) closed set
   (c) semi–open set (d) none of these

2. The complement of the unit circle $|z| = 1$ is
   (a) an open set (b) closed set
   (c) neither open nor closed (d) none of these

3. An annulus $\rho_1 < |z-a| < \rho_2$ is
   (a) connected (b) disconnected
   (c) semi connected (d) none of these

4. The principal value of $(i)^{-2i}$ is
   (a) $e^{-\pi}$ (b) e^{π}
   (c) $e^{-\pi/2}$ (d) $e^{-2\pi}$

5. Principal value of $\log i$ is
   (a) $\frac{i\pi}{2}$ (b) $-\frac{i\pi}{2}$
   (c) $\frac{\pi}{2}$ (d) $\frac{i}{2}$

6. The real part of exp (exp $i\theta$) is
   (a) $e^{\cos\theta}$ (b) $e^{\cos\theta}\sin(\sin\theta)$
   (c) $e^{\cos\theta}\cos(\sin\theta)$ (d) $e^{\cos\theta}\cos(\cos\theta)$

7. The value of $\sin(\log i^i)$ is
   (a) 1 (b) –1
   (c) π (d) $-\pi$

8. The value of $(\sqrt{i} + \sqrt{-i})$ is
   (a) $\sqrt{2}$ (b) 0
   (c) i (d) $-i$

COMPLEX ANALYSIS

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

9. *Which of the following(s) is/are false, when* $z = x + iy$?
   (a) amp $z = -$amp $\bar{z}$
   (b) amp $z \neq -$amp $\bar{z}$
   (c) amp $z \geq$ amp $\bar{z}$
   (d) amp $z \leq$ amp $\bar{z}$

10. *Imaginary part of* $\sin \bar{z}$ *is not*
   (a) cos x coshy
   (b) sinx sinhy
   (c) cosx sinhy
   (d) –cosx sinhy

11. *If* $i^{i^{\cdots}} = \alpha + i\beta$, *then which of the following is/ are true?*
   (a) $\dfrac{\alpha}{\beta} = \tan\dfrac{\pi\alpha}{2}$
   (b) $\alpha^2 + \beta^2 = e^{-\pi\beta}$
   (c) $\alpha^2 + \beta^2 = e^{\pi\beta}$
   (d) $\dfrac{\beta}{\alpha} = \tan\dfrac{\pi\alpha}{2}$

ASSIGNMENT - 1.2

NOTE: CHOOSE THE BEST OPTION

1. If $\log_{0.3}|z-1| > \log_{0.3}|z-i|$, then
(a) $x+y<0$ (b) $x-y>0$ (c) $x+y>0$ (d) $x-y<0$

2. Real part of a^i, where a is real, is $(a>0)$
(a) $e^{-2n\pi}\cos(\log_e a)$
(b) $e^{n\pi}\cos(\log_e a)$
(c) $\cos a$
(d) $e^{-2n\pi}(\log_e a)$

3. Real part of the principal value of $i^{\log(1+i)}$ is
(a) $e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$
(b) $e^{\frac{\pi^2}{8}}\log 2$
(c) $e^{\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$
(d) $e^{\frac{\pi^2}{8}}$

4. If $i^{i^i} = \cos\theta + i\sin\theta$, then the value of θ is
(a) $\left(2m+\frac{1}{2}\right)\pi e^{-\left(2n+\frac{1}{2}\right)\pi}$
(b) $\left(2m-\frac{1}{2}\right)\pi e^{-\left(2n+\frac{1}{2}\right)\pi}$
(c) $\left(2m+\frac{1}{2}\right)\pi e^{-\left(2n-\frac{1}{2}\right)\pi}$
(d) $\left(2m+\frac{1}{2}\right)\pi e^{+\left(2n-\frac{1}{2}\right)\pi}$

5. The function $e^z - \sinh z$ is
(a) even
(b) odd
(c) neither even nor odd
(d) the one about which nothing can be said

6. Given that the equation $z^2+(p+iq)z+r+is=0$, where p, q, r and s are non-zero, has a real root. Then
(a) $pqr = r^2+p^2s$ (b) $prs = q^2+r^2p$ (c) $qrs = p^2+s^2q$ (d) $pqs = s^2+q^2r$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

7. The value of $e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}$ at the point $x=\frac{\pi}{4}$ is not
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) 2

8. $|z-1| = |z+i|$ represents a/an
(a) infinite set (b) straight line (c) unbounded set (d) convex set

ASSIGNMENT - 1.3

NOTE: CHOOSE THE BEST OPTION

1. Let $z_1 = -z$, $z_2 = iz$, $z_3 = z - iz$. The area of the triangle in the Gaussian plane is
(a) $\frac{3}{2}|z|^2$ (b) $2|z|^2$ (c) $\frac{1}{2}|z|^2$ (d) none of these

2. The maximum value of $|z|$, when z satisfies the condition $\left|z-\frac{2}{z}\right| = 2$ is
(a) $\sqrt{3}-1$ (b) $\sqrt{3}$ (c) $\sqrt{3}+1$ (d) $\sqrt{2}+\sqrt{3}$

3. If $f(z) - z^2 + 2$, then the minimum value of $|f(z)|$ over the closed region $|z|\le 1$ is
(a) 0 (b) 1 (c) 3 (d) 2

4. The real part of the complex number $(1+i)^n$ is
(a) $2^{\frac{n}{2}} \cos\frac{n\pi}{4}$ (b) $2^n \cos\frac{n\pi}{2}$ (c) $2^{-\frac{n}{2}} \cos n\pi$ (d) $2^n \cos\frac{n\pi}{2}$

5. If $\left|\frac{z-2}{z-1}\right| = 2$, then Re(z) is
(a) $\frac{3}{4}|z|^2$ (b) $\frac{4}{3}|z|^2$ (c) $\frac{3}{4}|z|$ (d) $\frac{3}{4}|z-1|^2$

6. i^i forms a geometric progression whose common ratio is
(a) $e^{-2\pi}$ (b) $e^{2\pi}$ (c) e^{π} (d) $\frac{\pi}{2}$

7. The value of i^i is
(a) real (b) imaginary
(c) real and imaginary (d) 1

8. Equation of the circle described on the join of the points a, b as diameter is
(a) $2z\bar{z}-(\bar{a}+\bar{b})z-(a+b)\bar{z}+(a\bar{b}+\bar{a}b)=0$
(b) $2z\bar{z}+(\bar{a}+\bar{b})z+(\bar{a}+\bar{b})\bar{z}+(a\bar{b}-\bar{a}b)=0$
(c) $z\bar{z}-(a+b)z+(a-b)\bar{z}+(a\bar{b}-\bar{a}b)=0$
(d) $z\bar{z}-(\bar{a}+\bar{b})z-(a+b)\bar{z}+(a\bar{b}-\bar{a}b)=0$

9. $|z-4i|^2+|z+4i|^2 = 10$ will represent
(a) circle (b) ellipse (c) hyperbola (d) none of these

10. *For any complex number z, the minimum value of* $|z| + |z - 1|$ *is*

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

11. *If the imaginary part of* $\frac{2z+1}{iz+1}$ *is* -2*, then the locus of a point representing z, is a*

(a) circle
(b) straight line
(c) parabola
(d) none of these

12. *The equation* $\tan z = z$ *has*

(a) only real roots
(b) purely imaginary roots
(c) complex roots with non-zero real part
(d) No roots

13. *If* $2\cos\alpha_1 = a + \frac{1}{a}$, $2\cos\alpha_2 = b + \frac{1}{b}$ *etc., then* $abc\ldots + \frac{1}{abc\ldots}$ *will be given by*

(a) $\cos(2\alpha_1 + 2\alpha_2 + \ldots)$
(b) $2\cos(\alpha_1 + \alpha_2 + \ldots)$
(c) $2\sin(\alpha_1 + \alpha_2 + \ldots)$
(d) $\sin(2\alpha_1 + 2\alpha_2 + \ldots)$

14. *If* α, β *are roots of the equation* $x^2 - 2x + 4 = 0$, $\alpha^n + \beta^n$ *is equal to*

(a) $2^{n+1}\cos\left(\frac{n\pi}{3}\right)$
(b) $2^n\cos\left(\frac{n\pi}{3}\right)$
(c) $2^{n+1}\sin\left(\frac{n\pi}{3}\right)$
(d) $2^n\sin\left(\frac{n\pi}{3}\right)$

15. *The real part of the principal value of* 4^{4-i} *is*

(a) $256\cos(\ell n\ 4)$
(b) $64\cos(\ell n\ 4)$
(c) $16\cos(\ell n\ 4)$
(d) $4\cos(\ell n\ 4)$

16. *A function* $f(z) = z^2$ *have real and imaginary parts as*

(a) $Re(f(z)) = x^2$, $Im(f(z)) = y^2$
(b) $Re(f(z)) = x^2 + y^2$, $Im(f(z)) = 2xy$
(c) $Re(f(z)) = x^2 - y^2$, $Im(f(z)) = 2xy$
(d) $Re(f(z)) = x^2 + y^2$, $Im(f(z)) = x^2 - y^2$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

17. *If* $z = x + iy$*, then the equation* $z^2 = \bar{z}$ *has*

(a) only one solution
(b) two non-zero solutions
(c) four solutions
(d) three non-zero solutions

18. *The complex numbers* $z_1 = 1 + 2i$, $z_2 = 4 - 2i$ *and* $z_3 = 1 - 6i$ *does not form the vertices of a / an*

(a) right angled triangle
(b) isosceles triangle
(c) equilateral triangle
(d) scalene triangle

19. *For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the value of $|z_1 - z_2|$ has*

(a) minimum value zero
(b) minimum value two
(c) maximum value 22
(d) maximum value 17

20. *The inequality $|z-4|<|z-2|$ is satisfied by all points in the region*

(a) $Re(z) > 3$
(b) $Re(z) > 5$
(c) $Re(z) > 4$
(d) $Re(z) < 3$

21. *z is any non-zero complex number. Which of the following is/are false?*

(a) $\left|\frac{z}{|z|} - 1\right| \leq |Arg\, z|$
(b) $\left|\frac{z}{|z|} + 1\right| \leq |Arg\, z|$
(c) $\left|\frac{z}{|z|} - 1\right| \geq |Arg\, z|$
(d) $\left|\frac{z}{|z|} - 1\right| > |Arg\, z|$

22. *The closure of the set $\{z: 0 < Arg\, z < \pi/2\}$ is*

(a) $0 \leq Arg\, z \leq \frac{\pi}{2}$
(b) $0 \leq Arg\ z \leq \frac{\pi}{2} \cup \{0\}$
(c) $\{(x, y): x \geq 0, y \geq 0\}$
(d) $\{(x, y): x > 0, y > 0\}$

23. *If $|z + \bar{z}| = |z - \bar{z}|$, then locus of z is not*

(a) a pair of straight lines
(b) a rectangular hyperbola
(c) a line
(d) an ellipse

24. *The set of limit points of $\left\{z : z = \frac{1}{m} + i\frac{1}{n}, n \in \mathbb{Z}\right\}$ is given by*

(a) $\{0\}$
(b) $\{z \in \mathbb{C} : |z + \pi| = 2\}$
(c) $\{0, \frac{1}{n}, \frac{i}{n}, n \in \mathbb{Z}\setminus\{0\}\}$
(d) all of these

25. $Log(-2+2i)^2$ *is*

(a) $2[\log(-1 + i) + \log 2]$
(b) $\ln 8 - i\frac{\pi}{2}$
(c) $2\log(-2 + 2i)$
(d) none of these

26. *The principal value of $(1 + \sqrt{3}i)^i$ is*

(a) $i \log(1 + \sqrt{3}i)$
(b) $e^{i\ln 2 - \pi/3}$
(c) $e^{i\ln 2 - i\pi/6}$
(d) none of these

27. $\log(-3+3i)^2$ is
(a) $2\log(-3+3i)$
(b) $\ln 18 - i\pi/2$
(c) $2[\log 9 + \log(-1+i)]$
(d) none of these

28. Real part of $\log_e(1+i\tan\alpha)$ is
(a) $\log_e \sec\alpha$
(b) $-\log_e \sec\alpha$
(c) $\log_e \tan\alpha$
(d) $\log_e i$

29. If m and n are integers, then the value of the complex number $\log_i i$ is given by
(a) $\frac{4m+1}{4n+1}$
(b) $e^{\frac{4m+1}{4n+1}}$
(c) $\log\frac{4m+1}{4n+1}$
(d) 1

30. Which of the following is / are true?
(a) $Re(e^z)=e^x\cos y$
(b) $Im(e^z)=e^x\sin y$
(c) $Re(e^z)=e^y\cos x$
(d) $Im(e^z)=e^y\sin x$

31. A function $f(z)=\cos z$, has real and imaginary parts as
(a) $Re(f(z))=-\cos x\cosh y$,
(b) $Im(f(z))=\sin x\sinh y$
(c) $Re(f(z))=\cos x\cosh y$,
(d) $Im(f(z))=-\sin x\sinh y$

32. A function $f(z)=\log z$ has
(a) $Re(\log z)=-\log r$
(b) $Im(\log z)=\theta+n\pi$
(c) $Re(\log z)=\log r$
(d) $Im(\log z)=\theta+2n\pi$

33. If the set S is open, then which of the following is not always true ?
(a) S does not contain its boundary points
(b) S contains its boundary points
(c) S have finite boundary points
(d) S contains all its limits points

ANSWERS TO EXERCISES

(PRACTICE SET)

Exercise 1: (c)

Exercise 2: (b,c)

Exercise 3: $\ell n 2+\frac{i\pi}{2}$

Exercise 4: $e^{\frac{1}{2}\ell n2-\frac{7\pi}{4}-2n\pi}\cos\left(\frac{7\pi}{4}+\frac{1}{2}\ell n2\right), n\in\mathbb{Z}$

Exercise 5: (c,d)

Exercise 6: (a)

Exercise 7: (d)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 1.1

1. (b) 2. (a) 3. (a) 4. (b) 5. (a) 6. (c) 7. (b)
8. (a)
9. (b,c,d) 10. (a,b,c) 11. (b,d)

ASSIGNMENT - 1.2

1. (b) 2. (a) 3. (a) 4. (a) 5. (a) 6. (d)
7. (a,b,d) 8. (a,b,c,d)

ASSIGNMENT - 1.3

1. (a) 2. (c) 3. (b) 4. (a) 5. (a) 6. (a) 7. (a)
8. (a) 9. (d) 10. (a) 11. (b) 12. (a) 13. (b) 14. (a)
15. (a) 16. (c)
17. (c,d) 18. (a,c,d) 19. (b,c) 20. (a,b,c) 21. (b,c,d) 22. (b,c) 23. (b,c,d)
24. (c) 25. (b) 26. (b) 27. (b) 28. (a) 29. (a) 30. (a,b)
31. (c, d) 32. (c,d) 33. (b,c,d)

CHAPTER - 2
ANALYTICITY & C-R EQUATIONS

INTRODUCTION:

The differentiability of a real valued function in real analysis has already been studied. In complex analysis, differentiability of a function also has the same meaning. But the analyticity of a function at a point, is different from differentiability. The analyticity of a function is something wider than differentiability. In this chapter, the conditions for a function to be analytic and the properties of an analytic function are discussed.

2.1. CONTINUITY

2.1.1. Limit of function:

Let $f: D \to \mathbb{C}$ *be any function in a domain D given by* $w = f(z) = u(x, y) + iv(x, y)$

Let $z_0 \in \mathbb{C}$ *be a limit point of D, then* l *is the limit of f at* $z=z_0$ *if for any given* $\in > 0$, *there exists a* $\delta > 0$ *such that* $| f(z) - l | < \in$, *whenever* $0 < | z - z_0 | < \delta$ *in D and we write* $\lim_{z \to z_0} f(z) = l$

Note: 1. $\lim_{z\to z_0} f(z)$ *exists iff* $\lim_{(x,y)\to(x_0,y_0)} u(x, y)$ *and* $\lim_{(x,y)\to(x_0,y_0)} v(x, y)$ *exists.*

2. *The limit of a function f(z), if exists, must be unique.*

2.1.2. Continuity at a Point:

A function $f(z)$ *is said to be continuous at* z_0 *if* $\lim_{z \to z_0} f(z) = f(z_o)$

A function $f(z)$ *is said to be continuous at any interior point* z_o *of region R if for all* $\in > 0$, *there exists* $\delta > 0$ *such that* $|f(z) - f(z_o)| < \in$, *whenever* $|z - z_o| < \delta$

2.1.3. Continuity in a Region:

A function f(z) is said to be continuous in a region if it is continuous at all points of the region.

Note: *(a) f(z) is a continuous function in a domain D, it means f(z) is continuous at every* $z \in D$.

(b) $f(z)=u(x, y) + iv(x, y)$ *is continuous function of z* $\Leftrightarrow$ *u and v are continuous functions of x and y.*

2.1.4. Theorems on Continuity:

Theorem 2.1.1. *If f(z) and g(z) are continuous at* $z = z_0$, *then functions f(z)g(z), f(z)–g(z), f(z)+g(z) and* $\frac{f(z)}{g(z)}$ *are continuous, the last only if* $g(z_0) \neq 0$. *Similar results hold for continuity in a region.*

For Example: *All polynomials,* e^z, *sin z and cos z are continuous functions in every finite region.*

Theorem 2.1.2. *If $w = f(z)$ is continuous at $z = z_0$ and $z = g(\varsigma)$ is continuous at $\varsigma=\varsigma_0$ and if $\varsigma_0 = f(z_0)$, then the function $w=g[f(z)]$, called a function of a function or composite function, is continuous at $z = z_0$. This is sometimes briefly stated as: A continuous function of a continuous function is continuous.*

Theorem 2.1.3. *If $f(z)$ is continuous in a closed region, then it is bounded in that region, i.e., there exists a constant M such that $|f(z)| < M$ for all points z of the region.*

Theorem 2.1.4. *If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in that region.*

2.2. UNIFORM CONTINUITY

Let $f(z)$ be continuous in a region S. Then by the definition of continuity of $f(z)$, for any $\epsilon>0$, there exists $\delta>0$ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$. Here δ in general depend on both ϵ and the particular point z_0. However, if there exist δ depending on ϵ only and not on z_0, then $f(z)$ is said to be uniformly continuous in S.

Definition: *A function $f(z)$ is said to be uniformly continuous in a region S if for any $\epsilon>0$, there exists $\delta>0$ such that $|f(z_1)-f(z_2)|<\epsilon$ whenever $|z_1-z_2|<\delta\ \forall\ z_1, z_2 \in S$.*
Here, the choice of δ is independent of z_1 and z_2 in S.

Theorem 2.2.1. *If $f(z)$ is continuous in a closed region, then it is uniformly continuous in that region.*

2.3. DERIVATIVES

If $f(z)$ is single-valued in some region R of the z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} \qquad ...(1)$$

provided that the limit exists finitely, independent of the manner in which $\Delta z \to 0$. In such case, we say that $f(z)$ is differentiable at z. Sometimes h is used instead of Δz. Although differentiability implies continuity, the reverse is not true.

2.4. L'HOSPITAL RULE

Let $f(z)$ and $g(z)$ be analytic in a region containing the point z_0 and suppose that $f(z_0)=g(z_0)=0$ but $g'(z_0)\neq 0$. Then L'Hospital rule states that $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}$

In case $f'(z_0) = g'(z_0) = 0$, the rule may be extended.

Example 2.4.1. *The $\lim_{z\to 0}\frac{\bar{z}}{z}$ is*

(a) 0 *(b) 1*
(c) 1/2 *(d) does not exist*

Solution: (d) *We have* $f(z)=\frac{\bar{z}}{z}=\frac{x-iy}{x+iy}=\frac{x^2-y^2-2ixy}{x^2+y^2}$

$\therefore u(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ *and* $v(x,y) = \dfrac{-2xy}{x^2+y^2}$

Approaching (0,0) along y=mx, we get

$\lim\limits_{x\to 0} u(x,mx) = \lim\limits_{x\to 0}\dfrac{x^2-m^2x^2}{x^2+m^2x^2} = \lim\limits_{x\to 0}\dfrac{1-m^2}{1+m^2}$ *which depends on m.*

Hence, $\lim\limits_{(x,y)\to(0,0)} u(x,y)$ *does not exist and similarly* $\lim\limits_{(x,y)\to(0,0)} v(x,y)$ *does not exist.*

Hence, $\lim\limits_{z\to 0}\dfrac{\bar{z}}{z}$ *does not exist.*

2.5. ANALYTIC FUNCTION

If the derivative f′(z) exists at all points z of a region R, then f(z) is said to be analytic in R and is referred to as an analytic function in R or a function analytic in R. The terms regular and holomorphic are sometimes used as synonyms for analytic.

A function f(z) is said to be analytic at a point z_0 if there exists a neighbourhood $|z-z_0|<\delta$ such that $f'(z)$ exists at all points of the neighbourhood.

Properties of Analytic Functions:

If f(z) and g(z) are two analytic functions in a domain D, then

(i) $f(z) \pm g(z)$

(ii) $f(z)\,.\,g(z)$

(iii) $\dfrac{f(z)}{g(z)}[g(z)\neq 0]$

(iv) kf(z) (k is any constant), are also analytic in D.

Note : *Let D be an open region. Then a function f(z) is analytic in D iff it is differentiable in D.*

Results:

1. *If f(z) = u + iv is an analytic function of z=x + iy and ψ is any function of x and y with differential coefficient of first two orders, then* $\left(\dfrac{\partial\psi}{\partial x}\right)^2+\left(\dfrac{\partial\psi}{\partial y}\right)^2=\left[\left(\dfrac{\partial\psi}{\partial u}\right)^2+\left(\dfrac{\partial\psi}{\partial v}\right)^2\right]|f'(z)|^2$ *and*

   $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)|f'(z)|^2$$

2. *If f(z) is analytic function of z, then* $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|\operatorname{Re} f(z)|^2=2|f'(z)|^2$

3. *An analytic function with constant modulus is constant*

4. *If f(z) = u + iv is analytic function of z=x + iy in any domain D, then*

   (a) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|u|^p=p(p-1)|u|^{p-2}|f'(z)|^2$

(b) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$

(c) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$

5. $|f'(z)|^2=\frac{\partial(u,v)}{\partial(x,y)}$ *(Jacobian of u and v with respect to x and y)*

2.5.1. Entire Function: *A complex function f is said to be entire if it is analytic in the whole complex plane. Thus, the sum and the product of two or more entire functions are also entire functions.*

2.5.2. Singular Points or Singularity: *If a function f(z) fails to be analytic at a point z_0 but in every neighborhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of f(z).*

2.6. CAUCHY-RIEMANN EQUATIONS

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ is analytic in a region R is that, in R, u and v satisfy the Cauchy-Riemann equations, i.e.,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad ...(2)$$

If the partial derivatives in (2) are continuous in R, then the Cauchy-Riemann equations are sufficient conditions for f(z) to be analytic in R.

Note:

(a) $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$, *and* $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}=\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}$

(b) *In particular, at the origin* $\left(\frac{\partial f}{\partial x}\right)_{(0,0)}=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$ *and* $\left(\frac{\partial f}{\partial y}\right)_{(0,0)}=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$

2.6.1. Polar form of C-R equations:

If $f(z) = u + iv$ is an analytic function and $z=re^{i\theta}$, where u, v, r, θ are all real, then the Cauchy Riemann equations are $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$

PRACTICE SET – I

Exercise 1. *If $u + iv = \log \sin (x + iy)$, then*

(a) $u=\frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y-\cos 2x)\right]$ (b) $u=\frac{-1}{2}\log\left[\frac{1}{2}(\cosh 2y-\cos 2x)\right]$

(c) $v=\tan^{-1}(\cos x\tanh y)$ (d) $v=\cot^{-1}(\cot x\tanh y)$

Exercise 2. *Let $f(z)=2z^2-1$. Then the maximum value of $|f(z)|$ is less than or equal to on the unit disc $D=\{z\in\mathbb{C}:|z|\le 1\}$* ***(GATE 2007)***

(a) 1 *(b) 2*

(c) 3 *(d) 4*

Exercise 3. *Polar form of C-R equations are*

(a) $\frac{\partial u}{\partial\theta}=\frac{1}{r}\frac{\partial v}{\partial r},\frac{\partial u}{\partial r}=r\frac{\partial v}{\partial\theta}$ (b) $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\frac{\partial u}{\partial\theta}=-r\frac{\partial v}{\partial r}$

(c) $\frac{\partial u}{\partial\theta}=r\frac{\partial v}{\partial r},\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}$ (d) $\frac{\partial u}{\partial r}=r\frac{\partial v}{\partial\theta},\frac{\partial u}{\partial\theta}=-r\frac{\partial v}{\partial r}$

Exercise 4. *Consider the functions $f(z) = x^2 + iy^2$ and $g(z) = x^2 + y^2 + ixy$. At $z = 0$*

(a) f is analytic but not g.
(b) g is analytic but not f.
(c) both f and g are analytic.
(d) neither f nor g is analytic.

Exercise 5. *At $z = 0$, the function $f(z)=z^2\bar{z}$*

(a) does not satisfy C-R equations *(b) satisfies C-R equations*

(c) is differentiable *(d) is analytic*

Exercise 6. *Which of the following functions is/are nowhere differentiable?*

(a) $f(z) = \cos y - i \sin y$ *(b) $f(z) = \cos y + i \sin y$*

(c) $f(z) = x^2 - y^2 + i2xy$ *(d) $f(z) = \text{Im}(z) + 2i\,\text{Re}(z)$*

Exercise 7. *In order that the function $f(z)=\frac{|z|^2}{2}$, $z\neq 0$ be continuous at $z = 0$, we should define $f(0)$ equal to*

(a) 2 *(b) -1* *(c) 0* *(d) 1*

2.7. *HARMONIC FUNCTIONS*

If the second order partial derivatives of u and v with respect to x and y exist and are continuous in a region R, then from (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-0$

A real valued function $\psi = \psi(x, y)$ of real variables x and y is said to be harmonic in a region S if it has continuous partial derivatives of second order and satisfies Laplace's equation in two variable, i.e., $\nabla^2\psi = 0$ throughout S, where ∇^2 is the second order differential operator known as Laplace operator given by $\nabla^2\psi = \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = \psi_{xx} + \psi_{yy}$.

Result: *Let $f(z)=u + iv$ be an analytic function of $z=x + iy$. If u be a harmonic function, then* $\dfrac{\partial^2 u}{\partial z\partial \bar{z}} = 0$

2.7.1. ***Harmonic Conjugate:*** *The function g is said to be a harmonic conjugate of h if g and h are harmonic and the first order partial derivatives of g and h satisfy the C-R equations.*

Note:

(a) *If $f(z)=u + iv$ is analytic in a domain D, then v is a harmonic conjugate of u. Conversely, if v is a harmonic conjugate of u in a domain D, then $f(z) = u + iv$ is analytic in D.*

(b) *Two functions u(x,y) and v(x,y) are harmonic conjugates of each other if and only if they are constants.*

(c) *If $f(z)=u + iv$ be an analytic function of $z = x+ iy$, then families of curves $u = c_1$, $v = c_2$ are orthogonal to each other, where c_1 and c_2 are constants, i.e., u and v satisfies*

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}=0$$

(d) *The harmonic conjugate v is unique, upto an additive real constant.*

(e) *The harmonic function need not to be analytic but converse holds.*

2.8. *CONSTRUCTION OF AN ANALYTIC FUNCTION*

Method 1: Milne-Thomson's method.

We have $z = x + iy$ so that $x=\dfrac{z+\bar{z}}{2}$, $y=\dfrac{z-\bar{z}}{2i}$

$w = f(z) = u + iv = u(x, y) + iv(x, y)$ for $f(z) = u\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right) + iv\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)$

By setting $x=z$, $y=0$ so that $z=\bar{z}$ we obtain $f(z) = u(z,0) + iv(z,0)$

Also as $f'(z)=\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ (By Cauchy-Riemann Equations.)

Taking $\dfrac{\partial u}{\partial x} = \phi_1(x,y) = \phi_1(z,0)$, $\dfrac{\partial u}{\partial y} = \phi_2(x,y) = \phi_2(z,0)$ we get $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

Integration yields the result, $f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$, where c is a constant.

Thus, f(z) can be calculated directly if u is known. Similarly, if v(x, y) is given, then it can be proved that $f(z)=\int[\psi_1(z,0)+i\psi_2(z,0)]\,dz+c'$, *where* $\psi_1=\dfrac{\partial v}{\partial y}$, $\psi_2=\dfrac{\partial v}{\partial x}$

Formula for obtaining analytic function f(z) when real part of f(z) is given:

Let $f(z) = u(x,y) + i\,v(x,y)$ ...(1)

$\overline{f(z)} = u(x,y) - i\,v(x,y)$ ...(2)

Putting $x = 0,\ y = 0$ in (2), we get $\overline{f(0)} = u(0,0) - ic$ ...(3)

where ic is purely imaginary and c is real constant

Adding (1) and (2), we get $2u(x,y)=f(z)+\overline{f(z)}$

$2u(x,y) = f(x+iy) + \bar{f}(x-iy)$ [$\because$ for analytic function $\overline{f(z)} = \bar{f}(\bar{z})$] ...(4)

Replacing $x = z/2,\ y = z/2i$, (4) becomes $2u(z/2, z/2i) = f(z) + \overline{f(0)}$

Substituting value of $\overline{f(0)}$ from (3), we get $f(z) = 2u(z/2, z/2i) - \overline{f(0)}$

or $f(z) = 2u(z/2, z/2i) - u(0,0) + ic$

Note: *Function should be defined on (0, 0).*

Formula for obtaining analytic function f(z) when imaginary part of f(z) is given:

Subtracting (2) from (1), we get $f(z) - \overline{f(z)} = 2i\,v(x,y)$

$2i\,v(x, y) = f(x + iy) - \bar{f}(x - iy)$ ...(5)

Replacing $x=\dfrac{z}{2}$, $y=\dfrac{z}{2i}$ in equation (5), we get $2i\,v(z/2, z/2i) = f(z/2 + z/2i) - \bar{f}(0)$

$2i\,v(z/2, z/2 = f(z) - \bar{f}(0)$

$f(z) = 2i\,v(z/2, z/2i) + \bar{f}(0)$ ...(6)

we know from (2) that $\overline{f(z)} = u(x, y) - i\,v(x, y)$

$\therefore\ \bar{f}(0) = c - i\,v(0,0)$

$\therefore$ by (6), $f(z) = 2i\,v(z/2, z/2i) + c - i\,v(0,0)$

$f(z) = 2i\,v(z/2, z/2i) - i\,v(0,0) + c$

Examples of Method - I

Example 2.8.1. *Find the analytic function $f(z)=u + iv$ of which the real part $u = e^x(x\cos y - y\sin y)$. Or, Construct a function f(z) which has a real function $u(x, y)=e^x(x\cos y - y\sin y)$ as its real part, satisfying Laplace's equation.*

Solution: $\dfrac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$

$\frac{\partial u}{\partial y} = e^x[-x\sin y - \sin y - y\cos y]$

$\left(\frac{\partial u}{\partial x}\right)_{y=0} = e^x x + e^x = e^x(x+1)$, $\left(\frac{\partial u}{\partial y}\right)_{y=0} = e^x.0 = 0$

$\phi_1(x,0) = \left(\frac{\partial u}{\partial x}\right)_{y=0} = e^x\,(x+1)$

$\phi_2(x,0) = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$

By Milne's method, $f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\,dz + c,$

$\Rightarrow f(z) = \int\left[e^z(z+1) - i.0\right]dz + c = \int\left(ze^z + e^z\right)dz + c = (z-1)\,e^z + e^z + c = ze^z + c$

$\therefore f(z) = ze^z + c$

Method 2: *Suppose* $f(z) = u + iv$ *is analytic and u is known. To determine f(z). Firstly, we shall determine v.*

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left[\because\ df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right] = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$ *[By Cauchy Riemann equation]*

Taking $M = -\frac{\partial u}{\partial y}$, $N = \frac{\partial u}{\partial x}$, *we get* $dv = M\,dx + N\,dy$ ...(1)

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = -\nabla^2 u = 0$ *(As u satisfies Laplace's equation)*

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Consequently, (1) is exact differential equation. So (1) can be integrated and v can be determined. Now u and v are known and hence f(z) can be determined from the equation $f(z) = u + iv$

Examples of Method - II

Example 2.8.2. *Find the analytic function of which the real part is* $u = e^x(x\cos y - y\sin y)$

Solution: $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ ...(1)

$= -e^x(-x\sin y - y\cos y - \sin y)\,dx + e^x(x\cos y - y\sin y + \cos y)\,dy.$

Integrating (1), we get

$v = \int e^x\,(x\sin y + y\cos y + \sin y)dx$ *(treating y as constant)* $+ \int$ *(those terms which do not contain x)dy + c*

$= \sin y.\int xe^x dx + (y\cos y + \sin y)\int e^x dx + \int 0\,dy + c = [(x-1)\sin y + y\cos y + \sin y]\,e^x + c$

Thus $f(z) = u + iv$

$= e^x(x\cos y - y\sin y) + i\,[e^x(x\sin y + y\cos y) + c] = xe^x(\cos y + i\sin y) + iye^x(\cos y + i\sin y) + ic$

$= (x + iy)\,e^x.\,e^{iy} + ci = ze^z + ci$

Example 2.8.3. *Find all the harmonic functions of the form* $u=\phi\left(\sqrt{x^2+y^2}\right)$ *that are not constant, where the second order partial derivative of* ϕ *is continuous.*

Solution: $u=\phi\left(\sqrt{x^2+y^2}\right)$ ...(1)

Let $t=\sqrt{x^2+y^2}$,

$$\frac{\partial t}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\quad \frac{\partial t}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

Taking partial derivative of (1) w.r.t. x, $u_x=\phi'(t)\frac{x}{\sqrt{x^2+y^2}}=\frac{x\phi'(t)}{t}$

Also, $u_y=\dfrac{y\phi'(t)}{t}$

$$u_{xx}=\frac{\phi'(t)}{t}+x\left(\frac{t\phi''(t)-\phi'(t)}{t^2}\right)\left(\frac{x}{t}\right)$$

$$u_{yy}=\frac{\phi'(t)}{t}+y\left(\frac{t\phi''(t)-\phi'(t)}{t^2}\right)\left(\frac{y}{t}\right)$$

For harmonic function $u_{xx}+u_{yy}=0$

$$\Rightarrow \frac{2\phi'(t)}{t}+\frac{(x^2+y^2)}{t^2}\frac{[t\phi''(t)-\phi'(t)]}{t}=0$$

$$\Rightarrow \frac{2\phi'(t)}{t}+\phi''(t)-\frac{\phi'(t)}{t}=0$$

$$\Rightarrow \phi''(t)+\frac{\phi'(t)}{t}=0 \text{ or } \frac{d^2\phi}{dt^2}+\frac{1}{t}\frac{d\phi}{dt}=0$$

let $q=\dfrac{d\phi}{dt}$ and $\dfrac{dq}{dt}=\dfrac{d^2\phi}{dt^2}$, then $\dfrac{dq}{dt}+\dfrac{1}{t}q=0$ or $\dfrac{dq}{q}+\dfrac{dt}{t}=0$ or $\log q+\log t=\log A$

$$\therefore q=\frac{A}{t}$$

$$\therefore \frac{d\phi}{dt}=\frac{A}{t} \text{ or } d\phi=\frac{A}{t}dt$$

$\therefore \phi=A\log t+B$ or $\phi=A\log\left(\sqrt{x^2+y^2}\right)+B$

Example 2.8.4. For given $u(x,y)=x^3-3xy^2+3x^2-3y^2+1$. The analytic function f(z) is

(a) z^3-3z^2+c (b) z^3+3z^2+c (c) z^2+3z^3+c (d) z^2-3z^3+c

Solution: (b) We have $u=x^3-3xy^2+3x^2-3y^2+1$

$\therefore u_x=3x^2-3y^2+6x$ and $u_y=-6xy-6y$

$\therefore \phi_1(z,0)=u_x(z,0)=3z^2+6z$, $\phi_2(z,0)=u_y(z,0)=0$

By Milne's Method $f(z)=z^3+3z^2+c$

PRACTICE SET – II

Exercise 1. *The value of m for which the function* $f(x,y) = 2x - x^2 + my^2$ *may be harmonic is*

(a) 0 (b) 1 (c) 2 (d) 3

Exercise 2. *Which of the following functions is/are harmonic?*

(a) cosx coshy (b) $y^3 - 3x^2y$ (c) e^x cosy (d) e^x siny

Exercise 3. *If u + iv is analytic function, then dv is equal to*

(a) $\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial y}dy$ (b) $-\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy$ (c) $\frac{\partial u}{\partial x}dx - \frac{\partial u}{\partial y}dy$ (d) $\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy$

KEY POINTS

- *A function f(z) is said to be analytic at point* z_o *if there exists a neighborhood of* z_o *in which f(z) exists at all points*

- *A function is said to be entire if it is analytic in the whole complex plane.*

- *If a function f(z) is analytic in a domain D, then f(z)=u+iv satisfies Cauchy-Reimann Equations (or C-R equations) in D i.e* $u_x = v_y$ $u_y = -v_x$

- *If f(z)= u+iv is analytic in a domain D, then u and v are harmonic functions*

  i.e $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

  Also v is harmonic conjugate of $u \Rightarrow u_x = v_y$ *and* $u_y = -v_x$

- *If f(z)= u+iv is analytic, then the curves* $u=c_1$ *and* $v=c_2$ *cut each other at right angle. Here* c_1 *and* c_2 *are constants.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *Let f(z) be an entire function such that for some constant* α, $|f(z)| \le \alpha |z|^3$ *for* $|z| \ge 1$ *and* $f(z)=f(iz)$ *for all* $z \in \mathbb{C}$. *Then* **(GATE-2006)**

(a) $f(z) = \alpha z^3$ for all $z \in \mathbb{C}$ (b) f(z) is a constant

(c) f(z) is a quadratic polynomial (d) no such f(z) exists

Solution : (b) *Let f(z) be an entire function such that for some constant* α, $|f(z)| \le \alpha|z|^3$ *for* $|z| \ge 1$ *and* $f(z)=f(iz)$, $\forall z \in \mathbb{C}$.

Consider $f(z) = a_0 + a_1z + a_2z^2 + a_3z^3$. As $f(z) = f(iz)$

$\Rightarrow a_0 + a_1z + a_2z^2 + a_3z^3 = a_0 + a_1iz - a_2z^2 - a_3iz^3$

$\Rightarrow a_1(1-i)z + 2a_2 z^2 + (1+i)a_3 z^3 = 0$
$\Rightarrow a_1 = 0, a_3 = 0$ *and* $a_2 = 0$
So, $f(z) = a_0$, *i.e., constant*

Example 2. *Which of the following is not the real part of an analytic function?* *(GATE-2006)*

(a) $x^2 - y^2$

(b) $\frac{1}{1+x^2+y^2}$

(c) $\cos x \cosh y$

(d) $x + \frac{x}{x^2+y^2}$

Solution: (b) *If* $f(z) = u + iv$ *is analytic in a domain D, then both u and v are harmonic in D and* $u(x, y)$ *is harmonic in a domain D if all of its second order partial derivatives are continuous in D and* $u(x, y)$ *satisfies the Laplace equation,*

$$\text{i.e. } \nabla^2(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We will check by options
For option (d)

$$\text{let } u(x,y) = x + \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x}{(x^2+y^2)^2} - \frac{(y^2-x^2)\cdot 2\cdot 2x}{(x^2+y^2)^3}$$

$$= \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3} = \frac{-2x^3 - 2xy^2 - 4xy^2 + 4x^3}{(x^2+y^2)^3} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = -\frac{x\cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{2x}{(x^2+y^2)^2} + \frac{4xy\cdot 2y}{(x^2+y^2)^3} = \frac{-2x^3 - 2xy^2 + 8xy^2}{(x^2+y^2)^3} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}$$

$$\text{So, } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, $u(x,y) = x + \frac{x}{x^2+y^2}$ *can be real part of an analytic function. Similarly, we can check that functions given in option (a) and (c) are also harmonic.*

For option (b),

$$\text{let } u(x,y) = \frac{1}{1+x^2+y^2}$$

$$u_x = -\frac{2x}{(1+x^2+y^2)^2}$$

$$u_{xx} = -\frac{2}{(1+x^2+y^2)^2} + \frac{8x^2}{(1+x^2+y^2)^3} = \frac{-2(1+x^2+y^2)+8x^2}{(1+x^2+y^2)^3} = \frac{6x^2-2y^2-2}{(1+x^2+y^2)^3}$$

$$u_y = -\frac{2y}{(1+x^2+y^2)^2}$$

$$u_{yy} = -\frac{2}{(1+x^2+y^2)^2} + \frac{8y^2}{(1+x^2+y^2)^3} = \frac{-2(1+x^2+y^2)+8y^2}{(1+x^2+y^2)^3} = \frac{6y^2-2x^2-2}{(1+x^2+y^2)^3}$$

$$\Rightarrow u_{xx} + u_{yy} \neq 0 \text{ i.e., } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$$

$\therefore$ *u is not harmonic.*

$u(x, y) = \frac{1}{1+x^2+y^2}$ *cannot be real part of an analytic function.*

$\therefore$ *option (b) is correct.*

Example 3. *Define* $f:\mathbb{C}\to\mathbb{C}$ *by* $f(z) = \begin{cases} 0, & \text{if } \operatorname{Re}(z)=0 \text{ or } \operatorname{Im}(z)=0 \\ z, & \text{otherwise} \end{cases}$. *Then the set of points where f is analytic is* ***(GATE-2007)***

(a) $\{z : \operatorname{Re}(z) \neq 0 \text{ and } \operatorname{Im}(z) \neq 0\}$
(b) $\{z : \operatorname{Re}(z) \neq 0\}$
(c) $\{z : \operatorname{Re}(z) \neq 0 \text{ or } \operatorname{Im}(z) \neq 0\}$
(d) $\{z : \operatorname{Im}(z) \neq 0\}$

Solution: (a) $f(z) = \begin{cases} 0, & \text{if } \operatorname{Re}(z)=0 \text{ or } \operatorname{Im}(z)=0 \\ z, & \text{otherwise} \end{cases}$

Clearly $f(z)$ *is analytic, when* $\operatorname{Re}(z) \neq 0$ *and* $\operatorname{Im}(z) \neq 0$

If $\operatorname{Re}(z) = 0$, *i.e., y-axis*

Let $z = iy$

$f'(iy) = \lim_{z\to iy} \frac{f(z) - f(iy)}{z - iy} = 0$, *if we approach along the vertical line*

$[\because f(iy) = 0, f(z) = 0$ *along the vertical line*$]$

$= \lim_{z\to iy} \frac{z}{z - iy} = \infty$ *if we does not approach along the vertical line.*

Similarly if $Im(z) = 0$, *i.e., x - axis*

Let $z = x$

$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x} = 0$, *if we approach along the horizontal line*

$(\because f(x) = 0$ *and* $f(z) = 0$ *along the horizontal line*$)$

$= \lim_{z\to x} \frac{z}{z - x} = \infty$, *If we does not approach along horizontal line.*

Example 4. *Let S be the open unit disk and $f: S \to \mathbb{C}$ be a real valued analytic function with $f(0) = 1$. Then the set $\{z \in S: f(z) \neq 1\}$ is* *(GATE-2008)*

(a) empty
(b) nonempty finite
(c) countbly infinite
(d) uncountable

Solution: (a) *Let $f(z) = 1$, then f is real valued analytic function on S. Clearly $\{z \in S: f(z) \neq 1\} = \phi$.*
Hence, (b), (c), (d) are incorrect.
$\therefore$ *Option (a) is correct.*

Example 5. *Let $u(x,y) = 2x(1-y)$ for all real x and y. Then a function $v(x,y)$, so that $f(z) = u(x,y) + iv(x,y)$ is analytic, is* *(GATE-2010)*

(a) $x^2 - (y-1)^2$ (b) $(x-1)^2 - y^2$ (c) $(x-1)^2 + y^2$ (d) $x^2 + (y-1)^2$

Solution: (a) Given $u(x,y) = 2x(1-y) \quad \forall\, x, y \in \mathbb{R}$

$u_x = 2(1-y) = v_y$ ...(1)

Integrating with respect to y, we get

$\Rightarrow v = 2\left(y - \frac{y^2}{2}\right) + f(x) \quad \Rightarrow v_x = f'(x) = -u_y$

$\Rightarrow f'(x) = 2x \quad \Rightarrow f(x) = x^2 + c$

So, $f(z) = u + iv$

$f(z) = (2x - 2xy) + i(2y - y^2 + x^2 + c)$

Let $c = -1$

Then, $v = -(1-y)^2 + x^2 \quad \Rightarrow v = x^2 - (y-1)^2$

Example 6. *Let f(z) be analytic on $D = \{z \in \mathbb{C}: |z-1| < 1\}$ such that $f(1) = 1$. If $f(z) = f(z^2)$ for all $z \in D$, then which one of the following statement is not correct?* *(GATE-2010)*

(a) $f(z) = [f(z)]^2$ for all $z \in D$
(b) $f\left(\frac{z}{2}\right) = \frac{1}{2} f(z)$ for all $z \in D$
(c) $f(z^3) = [f(z)]^3$ for all $z \in D$
(d) $f'(1) = 0$

Solution: (b) *Take $f(z) = 1$ for all $z \in D$, then (b) option does not satisfy.*

Example 7. *Let f(z) be an entire function such that $|f(z)| \leq K|z|, \forall z \in \mathbb{C}$, for some $K > 0$. If $f(1) = i$, the value of f(i) is* *(GATE-2011)*

(a) 1 (b) -1 (c) i (d) $-i$

Solution: (b) *Defining $f(z) = iz$, then $|f(z)| \leq K|z|$ for $K = 1$*
Also, f(z) is an entire function.
$\therefore f(i) = -1$

$\Rightarrow f(z) = iz$
$\Rightarrow f(i) = i.i = i^2 = -1$

Example 8. *Let f be an entire function on $\mathbb{C}$ such that $|f(z)| \le 100 \log |z|$ for each z with $|z| \ge 2$. If $f(i)=2i$, then $f(1)$* **(GATE-2013)**

(a) must be 2
(b) must be 2i
(c) must be i
(d) cannot be determined from the given data

Solution: (b) *Defining $f(z)=2i$. Then f is an entire function on $\mathbb{C}$ and $|f(z)| \le 100 \log |z| \; \forall z$ such that $|z| \ge 2$*

$\therefore f(1)=2i$ *in this case*

Thus options (a), (c), (d) are incorrect

$\therefore$ *option (b) is correct.*

Example 9. *The function $f(z)=|z|^2+i\bar{z}+1$ is differentiable at* **(GATE-2014)**

(a) i (b) 1 (c) $-i$ (d) no point in $\mathbb{C}$

Solution: (c) $f(z)=|z|^2+i\bar{z}+1$

Put $z=x+iy \Rightarrow f(z)=x^2+y^2+i(\overline{x+iy})+1 = x^2+y^2+y+1+ix$

let $u=x^2+y^2+y+1$ and $v=x$

$u_x=2x,\ v_y=0,\ u_y=2y+1,\ v_x=1 \Rightarrow u_x=v_y$ and $u_y=-v_x \Rightarrow x=0$ and $y=-1$

$\therefore$ *Function is differentiable at $(0,-1)=-i$*

Example 10. *Let $u(x,y)=x^3+ax^2y+bxy^2+2y^3$ be a harmonic function and $v(x,y)$ its harmonic conjugate. If $v(0,0)=1$, then $|a+b+v(1,1)|$ is equal to* 10 **(GATE-2016)**

Solution: Given $u(x,y)=x^3+ax^2y+bxy^2+2y^3$

$u_x=3x^2+2axy+by^2,\ u_{xx}=6x+2ay$

$u_y=ax^2+2bxy+6y^2,\ u_{yy}=2bx+12y$

As, u is harmonic, therefore, $u_{xx}+u_{yy}=0$

$\Rightarrow 6x+2ay+2bx+12y=0 \Rightarrow (6+2b)x+(2a+12)y=0$

$\Rightarrow 6+2b=0 \Rightarrow b=-3$ & $2a+12=0 \Rightarrow 2a=-12 \Rightarrow a=-6$

As, $v(x,y)$ is the harmonic conjugate of $u(x,y)$

Therefore, $u_x=v_y$ & $u_y=-v_x \Rightarrow v_y=3x^2-12xy-3y^2$

Integrating w.r.t. y, we get

$$\Rightarrow v=3x^2y-12x\frac{y^2}{2}-\frac{3y^3}{3}+\phi(x)$$

$$\Rightarrow v=3x^2y-6xy^2-y^3+\phi(x) \qquad ...(1)$$

Differentiate w.r.t x, we get $v_x=6xy-6y^2+\phi'(x)$

Also, $v_x=6x^2+6xy-6y^2 \qquad ...(2)$

Comparing (1) and (2), we get $\phi'(x)=6x^2$

$\therefore \phi(x)=2x^3+c$, As $v(0,0)=1 \Rightarrow \phi(0)=1 \Rightarrow c=1$

Hence $\phi(x)=2x^3+1$

$\Rightarrow v = 3x^2y - 6xy^2 - y^3 + 2x^3 + 1$

$v(1, 1) = 3 - 6 - 1 + 2 + 1 = -7 + 6 = -1$

$|a + b + v(1, 1)| = |-6 - 3 - 1| = |-10| = 10$

Hence, Answer is 10.

Example 11. *Let* $f:\mathbb{C}\to\mathbb{C}$ *be a complex valued function given by* $f(z)=u(x,y)+iv(x,y)$. *Suppose that* $v(x,y)=3xy^2$. *Then* **(CSIR UGC NET JUNE-2011)**

(a) f *cannot be holomorphic on* $\mathbb{C}$ *for any choice of* u.
(b) f *is holomorphic on* $\mathbb{C}$ *for a suitable choice of* u.
(c) f *is holomorphic on* $\mathbb{C}$ *for all choices of* u.
(d) v *is not differentiable as a function of x and y.*

Solution: (a) *Given* $f:\mathbb{C}\to\mathbb{C}$ *is complex valued function such that* $f(z)=u(x,y)+iv(x,y)$ *and* $v(x,y)=3xy^2$. *A function* f *is holomorphic on* $\mathbb{C}$ *iff its real and imaginary part satisfies laplace equation (i.e. harmonic)*

$v_x = 3y^2, \quad v_y = 6xy$

$v_{xx} = 0, \quad v_{yy} = 6x$

$v_{xx} + v_{yy} \neq 0$

$\therefore$ f *cannot be holomorphic on* $\mathbb{C}$ *for any choice of* u

$\therefore$ *option (a) is correct.*

Example 12. *A bounded harmonic function in the unit disc centered at origin and taking the value* $\sin 2\theta$ *on the boundary is* **(CSIR UGC NET DEC-2012)**

(a) $r^2\sin 2\theta$ (b) $r\sin 2\theta$ (c) $\frac{1}{r}\sin 2\theta$ (d) $\frac{1}{r^2}\sin 2\theta$

Solution: (a) *We have to find bounded harmonic function in the unit disc centered at origin and taking value* $\sin 2\theta$ *on boundary*

We will check by options.

Clearly, $\frac{1}{r}\sin 2\theta$ *and* $\frac{1}{r^2}\sin 2\theta$ *are unbounded for* $r = 0$.

$\therefore$ *options (c) and (d) are incorrect.*

For option (a)

$f(r, \theta) = r^2\sin 2\theta$

Put $x = r\cos\theta$, $y = r\sin\theta$, *we get* $f(x, y) = 2xy$

$f_x = 2y, \quad f_{xx} = 0$

$f_y = 2x, \quad f_{yy} = 0$

$\Rightarrow f_{xx} + f_{yy} = 0$

Thus 'f' is harmonic

Also 'f' is bounded and take the value sin2θ on boundary of unit disc centered at origin.

Thus option (a) is correct

For option (b)

Take g(r, θ) = rsin2θ

Put x = rcosθ, y = rsinθ, we get $g(x, y) = \frac{2xy}{\sqrt{x^2+y^2}}$

It can be easily verified that 'g' is not harmonic

$\therefore$ *option (b) is incorrect.*

Example 13. *Let p(z), q(z) be two non - zero complex polynomials. Then* $p(z)\overline{q(z)}$ *is analytic if and only if* ***(CSIR UGC NET JUNE-2013)***

(a) p(z) is constant *(b) p(z)q(z) is constant*

(c) q(z) is a constant *(d)* $\overline{p(z)}q(z)$ *is a constant*

Solution: (c)

We will check by options

For option (a)

Take p(z) = z, non-constant,q(z)=1 but still $p(z).q(\overline{z}) = z$ *is analytic*

$\therefore$ *option (a) is incorrect*

With above example option (b) is also incorrect.

As $p(z)\ \overline{q(z)} = z$ *is analytic without being p(z) . q(z) , a constant*

$\Rightarrow$ *option (b) is also incorrect*

For option (d)

With above example $\overline{p(z)}\ q(z) = \overline{z}$, *non-constant.*

Thus, option (d) is also incorrect.

As Options (a), (b) and (d) are incorrect so remaining option i.e. option (c) is correct.

Example 14. *Let* $f: \mathbb{C} \to \mathbb{C}$ *be an analytic function. For z=x+iy, let* $u,v: \mathbb{R}^2 \to \mathbb{R}$ *be such that u(x,y)= Re f(z) and v(x,y) =Im f(z). Which of the following are correct?* ***(CSIR UGC NET JUNE-2013)***

(a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (b) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

(c) $\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0$ (d) $\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial x} = 0$

Solution: (a,b,c) $f: \mathbb{C} \to \mathbb{C}$ *is analytic function*

Let z = x + iy and f(z) = u + iv

As f is analytic

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial x \partial y} \qquad \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\Rightarrow$ *Options (a) and (b) are correct*

Now as $f(z)$ is analytic

$\Rightarrow$ *u and v have continuous second order derivatives and*

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ i.e., } \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\Rightarrow$ *Option (c) is also correct*

For option (d), As $\frac{\partial^2 v}{\partial x . \partial y} = \frac{\partial^2 v}{\partial y . \partial x}$

$$\Rightarrow \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial x} = 2\frac{\partial^2 v}{\partial x \partial y} \neq 0$$

$\Rightarrow$ *Option (d) is incorrect*

Example 15. *Let f be a non-constant holomorphic function in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ such that $f(0)=1$. Then it is necessary that* ***(CSIR UGC NET DEC 2013)***

(a) there are infinitely many points z in the unit disc such that $|f(z)|=1$

(b) f is bounded

(c) there are at most finitely many points z in the unit disc such that $|f(z)|=1$

(d) f is a rational function.

Solution: (a) *Take $f(z) = z + 1$*

Here $f(0) = 1$

Clearly, $f(z)$ is unbounded

$\therefore$ *option (b) is incorrect.*

Also $|f(z)| = |z + 1|$

$|f(z)| = 1 \Rightarrow |z + 1| = 1$, which represents a unit circle centered at $(-1, 0)$

$\therefore$ *There are infinitely many z which satisfies $|f(z)| = 1$*

Thus, option (a) is correct and (c) is incorrect

Further, take $f(z) = e^z$,

Here $f(0) = 1$ and $f(z)$ is analytic in the unit disc.

But $f(z)$ is not a rational function

$\therefore$ *option (d) is incorrect.*

Example 16. Let $u(x,y)=x^3-3xy^2+2x$. For which of the following functions v, u+iv is a holomorphic function on $\mathbb{C}$? **(CSIR UGC NET DEC-2014)**

(a) $v(x,y)=y^3-3x^2y+2y$

(b) $v(x,y)=3x^2y-y^3+2y$

(c) $v(x,y)=x^3-3xy^2+2x$

(d) $v(x,y)=0$

Solution: **(b)** Given $u(x,y)=x^3-3xy^2+2x$

For u + iv is holomorphic function

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

As, $\frac{\partial u}{\partial x}=3x^2-3y^2+2$ and $\frac{\partial u}{\partial y}=-6xy$

So we will check by options

For option (a),

$$\frac{\partial v}{\partial y}=3y^2-3x^2+2$$

$$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$$

$\therefore$ option (a) is incorrect

For option (b)

$$\frac{\partial v}{\partial y}=3x^2-3y^2+2, \frac{\partial v}{\partial x}=6xy$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial x}y=-\frac{\partial v}{\partial x}$$

$\Rightarrow$ option (b) is correct.

It can be easily verified that v(x, y) given in options (c) and (d), do not satisfy C-R. equations.

Example 17. Let f be a real valued harmonic function on $\mathbb{C}$, that is, f satisfies the equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$.

Define the functions

$$g=\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}$$

$$h=\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}$$

Then

(a) g and h are both holomorphic functions.

(b) g is holomorphic, but h need not be holomorphic.

(c) h is holomorphic, but g need not be holomorphic.

(d) both g and h are identically equal to the zero function.

Solution: *(b) Given* $g = \dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}$

$$u = \frac{\partial f}{\partial x}, v = \frac{-\partial f}{\partial y} \Rightarrow u_x = \frac{\partial^2 f}{\partial x^2} \quad v_y = \frac{-\partial^2 f}{\partial y^2}$$

$$\therefore u_x = v_y \quad \left[\because \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\right]$$

$$u_y = \frac{\partial f}{\partial y \partial x}, \quad v_x = \frac{-\partial f}{\partial x \partial y}$$

$\Rightarrow u_y = -v_x$

Also 'f' is harmonic function

$\Rightarrow$ *f has continuous partial second order derivatives.*

So g is holomorphic function.

$$h = \frac{\partial f}{\partial x} + \frac{i\partial f}{\partial y}, \quad u = \frac{\partial f}{\partial x}, v = \frac{\partial f}{\partial y}$$

$$u_x = \frac{\partial^2 f}{\partial x^2}, \quad v_y = \frac{\partial^2 f}{\partial y^2} = \frac{-\partial^2 f}{\partial x^2} \text{ as } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$\Rightarrow u_x \neq v_y$

So h is not a holomorphic function

Thus option (b) is correct.

ASSIGNMENT - 2.1

NOTE: CHOOSE THE BEST OPTION

1. *A function f(z+c)=f(z), where c is any number, then f is*
   (a) a periodic function
   (b) periodic function with period c
   (c) periodic function with period z
   (d) none of these

2. *The function $f(z) = u(z) + i\,v(z)$, $z \in \mathbb{C}$ is analytic iff*
   (a) u, v are harmonic functions
   (b) u, v are continuous functions
   (c) u, v satisfies Cauchy–Riemann equations
   (d) none of these

3. *A function $u = u(x, y)$ is harmonic if*
   (a) u have continuous second order derivative
   (b) u have continuous second order derivative and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
   (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
   (d) None of these

4. *If f and g are analytic functions, then*
   (a) f/g is always analytic
   (b) f/g is analytic whenever $g(z) \neq 0$
   (c) f/g is analytic whenever $f(z) \neq 0$
   (d) none of these

5. *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative. The given statement is*
   (a) True
   (b) False
   (c) True for z=0 only
   (d) True for $z=\infty$ only

6. *If u is a harmonic function, then* $\frac{\partial^2 u}{\partial z \partial \bar{z}} =$
   (a) 0
   (b) 1
   (c) 1/4
   (d) 1/2

7. *Let F(z) be defined in a domain D and differentiable at z = a, then* $F'(z) =$
   (a) $\lim_{z \to a} \frac{F(z) - F(a)}{z - a}$
   (b) $\lim_{z \to a} \frac{F(z + a) - F(a)}{z}$
   (c) $\frac{F(z + \Delta z) - F(z)}{\Delta z}$
   (d) $\lim_{z \to a} \frac{\Delta w}{\Delta z}$

8. *The function $f(z) = Re(z)$ is*
   (a) analytic
   (b) differentiable
   (c) continuous
   (d) none of these

9. *A derivative of branch of logarithm function is*
(a) z
(b) $1/z$
(c) 0
(d) none of these

10. *If G is open connected set in $\mathbb{C}$ and $f: G \to \mathbb{C}$ is a continuous function. Then f is a branch of logarithm if for $z \in G$*
(a) $z = \sin f(z)$
(b) $z = \cos f(z)$
(c) $z = \exp f(z)$
(d) $z = f(z)$

11. *If $G \subset \mathbb{C}$ and G is open and connected. Also if f is a branch of $\log z$ on G, then the totality of branches of $\log z$ are the functions*
(a) $f(z)$
(b) $f(z) + 2n\pi i$
(c) $f(z)$ + constant
(d) none of these

12. *$f(z) = |z|^2$ is continuous everywhere but nowhere differentiable*
(a) except at origin
(b) except at $|z| = 1$
(c) in $\mathbb{C}$
(d) except at $z = 1$

13. *Sufficient condition for $f(z)$ to be analytic is*
(a) u_x, u_y, v_x, v_y exist, continuous and satisfy C-R equations
(b) u_x, u_y, v_x, v_y exist
(c) C-R equations are satisfied
(d) $f(z)$ is single valued

14. *Polar form of C-R equations, is*
(a) $ru_r = v_\theta;\ u_\theta = -rv_r$
(b) $u_r = rv_\theta;\ ru_\theta = -v_r$
(c) $u_r = -rv_\theta;\ u_\theta = rv_r$
(d) $u_\theta = v_\theta;\ u_r = rv_r$

15. *The function $f(z) = |z|^2$ is*
(a) everywhere analytic
(b) nowhere analytic
(c) analytic at $z = 0$
(d) none of these

16. *The function $f(z) = Re(z^2)$ is*
(a) analytic
(b) continuous
(c) not continuous
(d) none of these

<u>NOTE: MORE THAN ONE OPTION MAY BE CORRECT</u>

17. *If a function $f(z)$ is analytic at a point $z = z_0$, then which of the following statements is/are true?*
(a) $f(z)$ is differentiable at z_0
(b) $f(z)$ is not defined at z_0,
(c) $f(z)$ is differentiable at infinitely many points
(d) $f(z)$ is continuous at z_0

18. *If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, then*
   (a) *f cannot be discontinuous at a*
   (b) *f is continuous at a*
   (c) *f is constant function*
   (d) *none of these*

19. *If f is analytic function in some domain, then in that domain*
   (a) *f is continuous*
   (b) *f is differentiable*
   (c) *f is not continuous*
   (d) *f is not differentiable*

20. *If a function f(z) is continuous at $z = z_0$, then which of the following statements hold true?*
   (a) *f is necessarily analytic at z_0*
   (b) *f is defined at z_0*
   (c) $\lim_{z \to z_0} f(z)$ *exist at z_0*
   (d) *f is necessarily differentiable*

21. *The function $w = e^z$ is*
   (a) *entire*
   (b) *periodic*
   (c) *not entire*
   (d) *not periodic*

22. *If G is either a whole plane $\mathbb{C}$ or some open disk and $u : G \to \mathbb{R}$ is harmonic function, then*
   (a) *u has a harmonic conjugate*
   (b) *u has no harmonic conjugate*
   (c) *u is not analytic*
   (d) *none of these*

23. *Let v be the harmonic conjugate of u. Which statement is true?*
   (a) *If u is the harmonic conjugate of v, then $f(z) = u + iv$ is constant.*
   (b) *u is also harmonic conjugate of -v.*
   (c) *If u is also the harmonic conjugate of v, then v-u is a constant function.*
   (d) $dv = u_x\,dy - u_y\,dx$.

24. *Function w=sinz is*
   (a) *entire*
   (b) *bounded*
   (c) *unbounded*
   (d) *nowhere analytic*

25. *If a function f(z) is continuous at z_0, then which of the following is true?*
   (a) *f(z) is necessarily differentiable at z_0*
   (b) *f(z) is not necessarily differentiable at z_0*
   (c) *f(z) is necessarily analytic at z_0*
   (d) *f(z) is not necessarily analytic at z_0*

ASSIGNMENT - 2.2

NOTE: CHOOSE THE BEST OPTION

1. The function $f(z) = \begin{cases} \frac{\text{Im } z}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$ at $z = 0$ is

(a) not continuous (b) continuous (c) not defined (d) none of these

2. $f(z) = \begin{cases} (\text{Re } z^2)/|z|^2 , & z \neq 0 \\ 0 , & z = 0 \end{cases}$ at $z = 0$ is

(a) continuous (b) not defined (c) not continuous (d) none of these

3. If $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$ and G is open and connected, then

(a) f is constant function (b) f is an increasing function
(c) f is decreasing function (d) none of these

4. If $u - v = e^x (\cos y - \sin y)$, then $w = f(z)$ is

(a) $e^{-z} + c$ (b) $e^z + c$ (c) $\frac{1}{e^{1/z}} + c$ (d) e^{-z+c}

5. The harmonic conjugate of $u(x, y) = y^3 - 3x^2y$ is

(a) $-y^3 + 3x^2y + c$ (b) $y^3 - 3xy^2 + c$ (c) $x^3 - 3x^2y + c$ (d) $x^3 - 3xy^2 + c$

6. The analytic function $w(z)$ whose $Im (w(z)) = \frac{x+y}{x^2+y^2}$ is (C, a real constant)

(a) $\frac{-1+i}{z} + C$ (b) $\frac{1+i}{z} + C$ (c) $\frac{1-i}{z} + C$ (d) $\frac{-1-i}{z} + C$

7. The conjugate harmonic function v of $u = e^{-2xy} \sin (x^2 - y^2)$, where u is harmonic, is

(a) $v = -e^{-2xy} \cos (x^2 - y^2) + C$ (b) $v = e^{-2xy} \cos (x^2 - y^2) + C$
(c) $v = e^{+2xy} \cos (x^2 - y^2) + C$ (d) $v = -e^{-2xy} \sin (x^2 - y^2) + C$

8. The derivative of $(z+2i)(i-z)/(2z-1)$ at $z = i$ is

(a) $(-6+3i)/5$ (b) $\frac{6+3i}{5}$ (c) $\frac{6-3i}{5}$ (d) $-\frac{3}{5}(2+i)$

9. If the real part of $f(z)$ is $u(x, y) = 2xy + \cosh x \sin y$, and $f(0) = 0$, then the value of analytic function $f(z)$ is

(a) $-i [z^2 + \sin hz]$ (b) $[z^2 + \sin hz]$ (c) $[z^2 - \sin hz]$ (d) $-[z^2 - \sin hz]$

10. Function $f(z) = xy + iy$ is

(a) everywhere continuous and analytic (b) everywhere continuous but not analytic
(c) discontinuous but analytic everywhere (d) neither continuous nor analytic.

11. If $f(z) = z|z|^2$, then $f(z)$ is differentiable
(a) at all points z (b) at $z = 1$ (c) only for $z = 0$ (d) none of these

12. The function $f(z) = z^2$ is
(a) infinitely differentiable (b) finitely differentiable
(c) not differentiable (d) none of these

13. The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^{\bar{z}}$ is
(a) an entire function (b) analytic in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$
(c) analytic at the any point 0. (d) not analytic at any point

<u>***NOTE: MORE THAN ONE OPTION MAY BE CORRECT***</u>

14. The function $|\bar{z}|^2$ is
(a) analytic (b) differentiable at z=0
(c) not analytic (d) continuous

15. If $f(z) = u + iv$ is a regular function in domain D, where $f(z) \neq 0$, the curves u = constant and v = constant.
(a) are not parallel (b) are perpendicular
(c) are inclined at 45^o (d) none of these

16. Consider the following limits
(i). $\lim\limits_{z \to 0} \frac{1}{z}$ (ii) $\lim\limits_{z \to \infty} \frac{1}{z}$ (iii) $\lim\limits_{z \to 0} \frac{1}{z^2}$ (iv) $\lim\limits_{z \to \infty} \frac{1}{z^2}$.
Which of the following statements are true?
(a) all the limit exists (b) except I and III the limit exists
(c) I and III limit exist (d) neither of the limits exist

17. If $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic, then values of a,b,c,d are
(a) $c = 1, d = a, b = 1$
(b) $d = 2, a = 2, a = -2c, b = -d/2$
(c) $a = 2, b = -1, c = -1, d = 2$
(d) because u, v are harmonic so a, b, c, d are not possible to find.

18. If $f(z) = z \operatorname{Re} z$, then $f(z)$ is not differentiable
(a) at all points z except z=0 (b) only for $z = 0$
(c) at $z = 1$ (d) at $z = 4$

19. If $f(z) = z \operatorname{Re} z$, then $f'(0)$ is not
(a) zero (b) one (c) three (d) four

ASSIGNMENT - 2.3

NOTE: CHOOSE THE BEST OPTION

1. $f(x,y) = e^y(\cos x + i\sin x)$ is
(a) an entire function
(b) analytic in $x^2+4y^2<24$
(c) nowhere analytic
(d) none of these

2. The function $w = \cos z$ is
(a) unbounded and entire
(b) bounded and entire
(c) bounded but nowhere analytic
(d) unbounded but nowhere analytic

3. $e^{\alpha x}\cos\beta y$ is a harmonic function
(a) for all α, β (b) if $\alpha^2+\beta^2=0$ (c) if $\alpha^2-\beta^2=0$ (d) none of these

4. u(x, y) = sinh x siny is harmonic in a domain D, then its harmonic conjugate in D is
(a) cosh x cosy+c (b) –cosh x cosy+c (c) sinh x cosy+c (d) cosh x siny+c

5. If an analytic function f(z) is such that Re {f '(z)} = 2y and f(1 + i) = 2, then the imaginary part of f(z) is
(a) $-2xy$ (b) x^2-y^2 (c) $2xy$ (d) y^2-x^2

6. Consider a function f(z) = u + i v defined on |z – i | < 1 , where u, v are real valued function of x, y, then f(z) is analytic for u equals to
(a) x^2+y^2 (b) $\ell n\,(x^2+y^2)$ (c) e^{xy} (d) $e^{x^2-y^2}$

7. If $f(z) = |z|^2$, then $f'(0)$ is
(a) 0 (b) 1 (c) does not exist (d) none of these

8. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \begin{cases} (\bar{z})^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$. Then f
(a) is not continuous at z = 0
(b) is differentiable but not analytic at z = 0
(c) is analytic at z = 0
(d) satisfies the Cauchy-Riemann equations at z=0

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

9. Let f(z)= |z|/Re(z) or 0 according as Re(z)≠0 or Re(z) = 0, then at z = 0 function f(z) is:
(a) having non zero limit (b) not differentiable (c) not continuous (d) continuous

10. If $f(z)=\begin{cases} \dfrac{1}{z}, & z \neq 0 \\ \infty, & z = 0 \\ 0, & z = \infty \end{cases}$. Then f(z) is
(a) discontinuous
(b) continuous
(c) analytic in the entire complex plane
(d) not differentiable at 0

ANSWERS TO EXERCISES

(PRACTICE SET – I)

Exercise 1: *(a,c)*
Exercise 2 *(c,d)*
Exercise 3: *(b)*
Exercise 4: *(d)*
Exercise 5: *(b,c)*
Exercise 6: *(a,d)*
Exercise 7: *(c)*

(PRACTICE SET – II)

Exercise 1: *(b)*
Exercise 2 *(a,b,c,d)*
Exercise 3: *(b)*

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 2.1

1. (b)	2. (d)	3. (b)	4. (b)	5. (a)	6. (a)	7. (a)
8. (c)	9. (b)	10. (c)	11. (b)	12. (a)	13. (a)	14. (a)
15. (b)	16. (b)					
17. (a,c,d)	18. (a,b)	19. (a,b)	20. (b,c)	21. (a,b)	22. (a)	23. (a,b,c,d)
24. (a,c)	25. (b,d)					

ASSIGNMENT - 2.2

1. (a)	2. (c)	3. (a)	4. (b)	5. (d)	6. (a)	7. (a)
8. (a)	9. (a)	10. (b)	11. (c)	12. (a)	13. (d)	
14. (b,c,d)	15. (a,b)	16. (a,c)	17. (b,c)	18. (a,c,d)	19. (b,c,d)	

ASSIGNMENT - 2.3

1. (c)	2. (a)	3. (c)	4. (b)	5. (d)	6. (b)	7. (a)
8. (d)						
9. (b,c)	10. (b,d)					

CHAPTER – 3
COMPLEX INTEGRATION

INTRODUCTION

In real analysis, Integration of a function depends on end points only, whereas in complex analysis integration depends on path. But if a function is analytic then it is independent of path. In complex analysis we have various tools (methods) to integrate a function. In this chapter we will discuss about different methods of finding complex integral and other important properties of an analytic function.

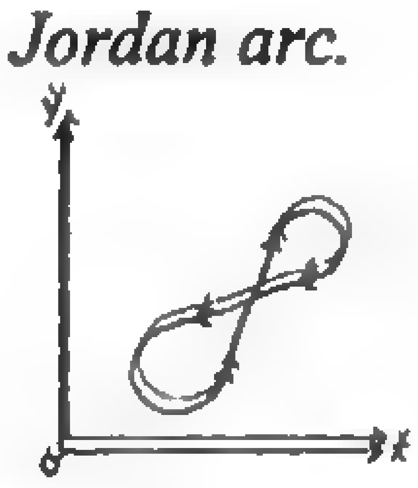
3.1. SOME DEFINITIONS

(1) ***Continuous Arc:*** *A continuous curve (or simply a curve or a path) in $\mathbb{C}$ is a continuous mapping (gamma) γ from a closed interval $[a, b]$, $a < b$, $a, b \in \mathbb{R}$ into $\mathbb{C}$. Here the points $\gamma(a)$ and $\gamma(b)$ are called initial and terminal points of the curve, respectively. A parametric representation of a continuous curve γ is given by $\gamma(t) = \phi(t) + i\psi(t)$, $t \in [a, b]$, where $\phi(t)$ and $\psi(t)$ are continuous real-valued functions on $[a,b]$*

(2) ***Multiple Points:*** *If $z = z(t) = \phi(t) + i\psi(t)$, when $x=\phi(t)$ and $y=\psi(t)$, is satisfied by more than one value of t in the given range, then the point z, or say the point (x,y) is called a multiple point of the arc.*

(3) ***Jordan Arc:*** *A continuous arc without multiple points is called a Jordan arc. Thus, for a point z on a Jordan curve, $z = \phi(t) + i\psi(t)$ is expressed as a single valued and $\phi(t), \psi(t)$ are continuous real valued functions on [a,b]. In addition, if $\phi'(t)$, $\psi'(t)$ are also continuous in the range $a \le t \le b$, then the arc is called a regular arc or a Jordan curve. A continuous Jordan curve consists of a chain of finite number of continuous arcs.*

For example: *$\gamma(t) = e^{it}$, $t \in [0, \pi]$, is a Jordan arc.*



(4) ***Contour:*** *A Jordan curve consisting of continuous chain of a finite number of regular arcs is known as contour. If A is the starting point of the first arc and B the end point of the last arc, then integral along such a curve is written as $\int_{AB} f(z)dz$. If the starting point A coincides with the end point B of the last arc, then the contour AB is a piecewise smooth curve and is called a closed contour..*

(5) ***Piecewise smooth arc (Contour):*** *A path γ is said to be piecewise smooth if there exists a partition P of [a, b] such that $a = t_0 < t_1 < \ldots < t_n = b$ and γ is smooth on each sub-interval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.*

(6) ***Simple curve:*** *A curve γ defined on $[a,b]$ is called simple if it does not intersect with itself.*

(7) **Rectifiable curve:** *A curve γ is rectifiable if it has finite arc length. Every piecewise smooth arc is rectifiable.*

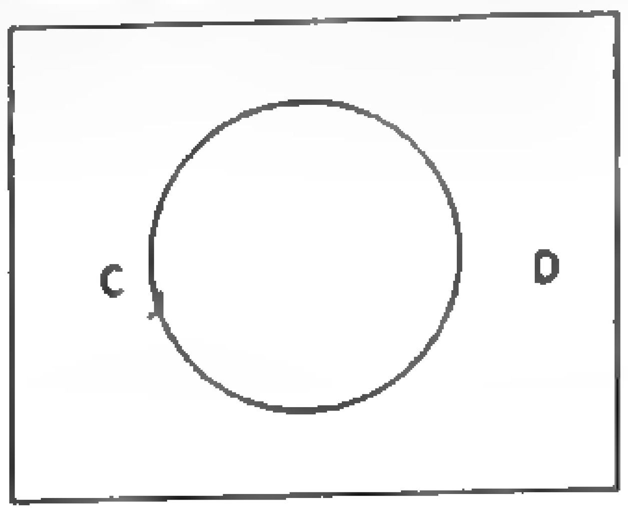
Note: (1) *The rectifiability and length of a curve are independent of its orientation.*

(2) *An arc $z = z(t) = x(t) + i\,y(t),\ a \le t \le b$ is rectifiable if and only if the functions x(t) and y(t) are of bounded variation in [a,b]. Length of arc z(t) is given by* $L = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}\,dt$

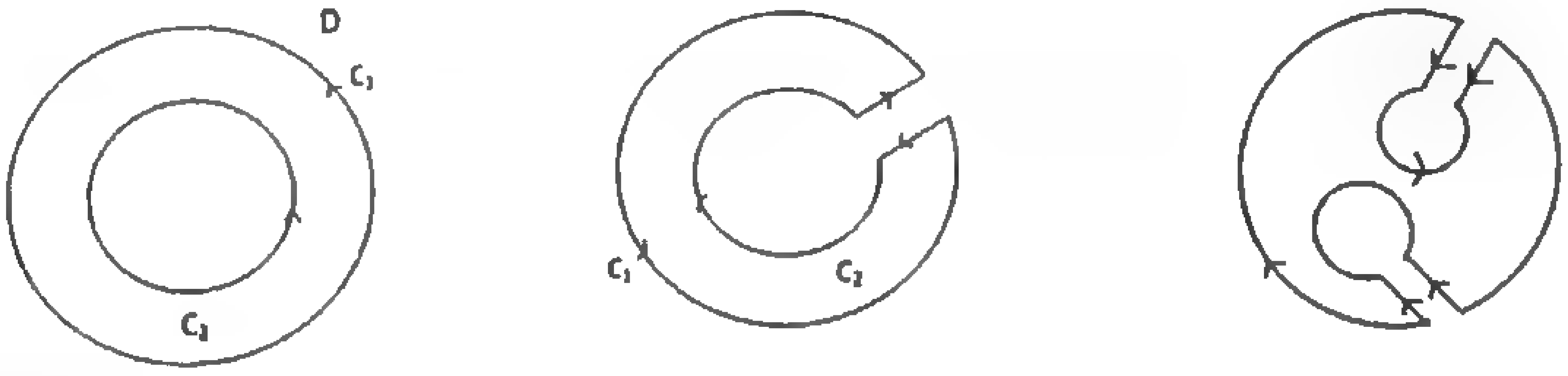
or $L = \int_a^b |z'(t)|\,dt$, *where the real-valued function* $|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ *is integrable over [a,b].*

(8) **Jordan Curve Theorem:** *A Jordan curve is the common boundary of the two regions into which the Argand plane is divided by it. Out of these two regions which are such that all points of it satisfy $|z|<M$, where M is some positive constant i.e. that region which is bounded is known as the interior or inside of the curve, while the other region is known as the exterior or outside of the curve.*

(9) **Simply Connected Region (or domain):** *A connected region or domain (D) is said to be a simply connected, if a closed curve C, contained in D has its interior contained in D, i.e., there is no hole.*



(10) **Multiple Connected Region:** *A region that is not simply connected, i.e., there are some holes is called multiple connected region. Multiple connected region is bounded by more than one curve.*
Any multiply connected region can be reduced to a simply connected region by giving it one or more cuts.



3.2. LINE INTEGRAL

For a real valued function f(x), definite integral $\int_a^b f(x)dx$ *is calculated always along the real axis (x-axis), from x=a to x=b i.e., path of integration is always along real axis.*

But for a complex function f(z), the path of the definite integral $\int_a^b f(z)dz$ *may be along any curve joining z=a to z=b i.e., value of the integral depends upon the path of integration.*

In case the initial point and final point coincide, then curve C, is a closed curve and integral is called contour integral and is denoted by $\oint_C f(z)dz$. *If* $f(z)=u(x,y)+i\,v(x,y)$, *where*

$z=x+iy \Rightarrow dz=dx+idy$ *and* $C:[a,b]\to\mathbb{C}$ *is closed contour, then*

$$\oint_C f(z)dz=\int_a^b (u+iv)(dx+idy)=\int_a^b (udx-vdy)+i\int_a^b (vdx+udy)$$

Therefore, definite integral of f(z) is reduced to two line integrals of real functions.

***Example:* 3.2.1.** *Evaluate* $\int_0^{1+i}(x^2-iy)dz$ *along* $y=x$.

Solution: *Along the line* $y=x \Rightarrow dy=dx$

Here, $z=x+iy=x(1+i)$ *and* $dz=dx\,(1+i)$

When $z=0 \Rightarrow x=0$ *and when* $z=1+i \Rightarrow x=1$

$\therefore$ *Given, integral is* $\int_0^1 (x^2-ix)(1+i)dx=(1+i)\left[\int_0^1 x^2dx-i\int_0^1 xdx\right]$

$$=(1+i)\left[\frac{x^3}{3}-i\frac{x^2}{2}\right]_0^1=(1+i)\left(\frac{1}{3}-\frac{i}{2}\right)=\frac{5-i}{6}$$

3.3. COMPLEX LINE INTEGRAL

Let f(z) be a continuous function of complex variable z defined on rectifiable arc L, then $\int_L f(z)dz$ *is called the complex line integral along rectifiable arc L or definite integral of f(z) from a to b along L is known as complex line integral and* $\int_L f(z)dz=\lim_{n\to\infty}\sum_{k=1}^{n}(z_k-z_{k-1})f(\xi_k)$, *where* ξ_k *is a point on each arc joining the point* z_{k-1} *to* z_k.

Properties of Complex Line Integral:

(i) $\int_L\{f(z)+\phi(z)\}dz=\int_L f(z)dz+\int_L\phi(z)dz$, *this property can be generalized for any finite number of functions.*

(ii) $\int_L f(z)dz=-\int_{-L}f(z)dz$, *where –L is the opposite arc of L.*

(iii) $\int_{L_1+L_2}f(z)dz=\int_{L_1}f(z)dz+\int_{L_2}f(z)dz$, *where the end point of* L_1 *coincides with the initial point of* L_2.

(iv) $\int_L cf(z)dz=c\int_L f(z)dz$, *where c is any complex constant.*

(v) $\int_L[c_1f_1(z)+c_2f_2(z)+\ldots+c_nf_n(z)]dz=c_1\int_L f_1(z)dz+c_2\int_L f_2(z)dz+\ldots+c_n\int_L f_n(z)dz$, *where* $c_1,c_2,\ldots,c_n$ *are complex constants.*

(vi) $\left|\int_L f(z)dz\right| \le \int_L |f(z)||dz|$

(vii) *If a function f(z) is continuous on a contour L of length l and if M is the upper bound of f(z) on L, i.e., $|f(z)| \le M$ on L, then* $\left|\int_L f(z)dz\right| \le Ml$

(viii) *The line integral* $\int pdx+qdy$, *defined in a domain D, depends only on the end points of* Γ *if and only if there exists a function U(x,y) in D such that* $\frac{\partial U}{\partial x}=p$ *and* $\frac{\partial U}{\partial y}=q$

For Example. *Evaluate* $\int_C (z^2+3z)dz$ *along the circle* $|z|=2$ *from (2,0) to (0,2).*

Solution: *We have,* $z=x+iy \Rightarrow dz=dx+idy$

$|z|=2 \Rightarrow \sqrt{x^2+y^2}=2 \Rightarrow x^2+y^2=4$

Put $x=2\cos t$, $y=2\sin t$, *t varies from 0 to* $\frac{\pi}{2}$.

$\therefore\ dx=-2\sin t\,dt,\ dy=2\cos t\,dt$

$z=2(\cos t+i\sin t)=2e^{it} \Rightarrow dz=(-2\sin t+2i\cos t)dt=2e^{i((\pi/2)+t)}dt$

$$\therefore \int_C (z^2+3z)dz=\int_0^{\pi/2}\left[(2e^{it})^2+3\cdot 2e^{it}\right]2e^{i((\pi/2)+t)}dt=2\int_0^{\pi/2}\left(4e^{2it}+6e^{it}\right)e^{i((\pi/2)+t)}dt$$

$$=8\int_0^{\pi/2}e^{i((\pi/2)+3t)}dt+12\int_0^{\pi/2}e^{i((\pi/2)+2t)}dt=\left[8\cdot\frac{e^{i((\pi/2)+3t)}}{3i}+12\,\frac{e^{i((\pi/2)+2t)}}{2i}\right]_0^{\pi/2}$$

$$=\frac{8}{3i}(e^{2\pi i}-e^{\pi i/2})+\frac{12}{2i}(e^{3\pi i/2}-e^{i\pi/2})=\frac{8}{3i}(1-i)+\frac{6}{i}(-i-i)=\frac{8}{3i}-\frac{8}{3}-12=-\frac{8i}{3}-\frac{44}{3}=-\left(\frac{44}{3}+\frac{8i}{3}\right)$$

3.3.1. Green's Theorem in the plane:

Let P(x,y) and Q(x,y) be continuous and have continuous partial derivatives in a region $\Re$ *and on its boundary C. Green's theorem states that* $\oint_C Pdx+Qdy=\iint_{\Re}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

The theorem is valid for both simply and multiple connected regions.

3.3.2. Complex form of Green's Theorem:

Let $F(z,\bar{z})$ *be continuous and have continuous partial derivatives in a region* $\Re$ *and on its boundary C, where* $z=x+iy, \bar{z}=x-iy$ *are complex conjugate coordinates. Then Green's theorem can be written in the complex form* $\oint_C F(z,\bar{z})dz=2i\iint_{\Re}\frac{\partial F}{\partial \bar{z}}dA$, *where dA represents the area of element dx dy.*

3.4. CAUCHY'S THEOREM

Statement:

If f(z) is analytic in a simply connected domain D and C is any closed curve inside, then $\oint_C f(z)dz=0$

Note: *Converse of Cauchy's theorem is not true, i.e., if* $\oint_C f(z)dz=0$ *for any closed curve C in D, then f(z) need not be analytic*

3.4.1. Some consequences of Cauchy's Theorem:

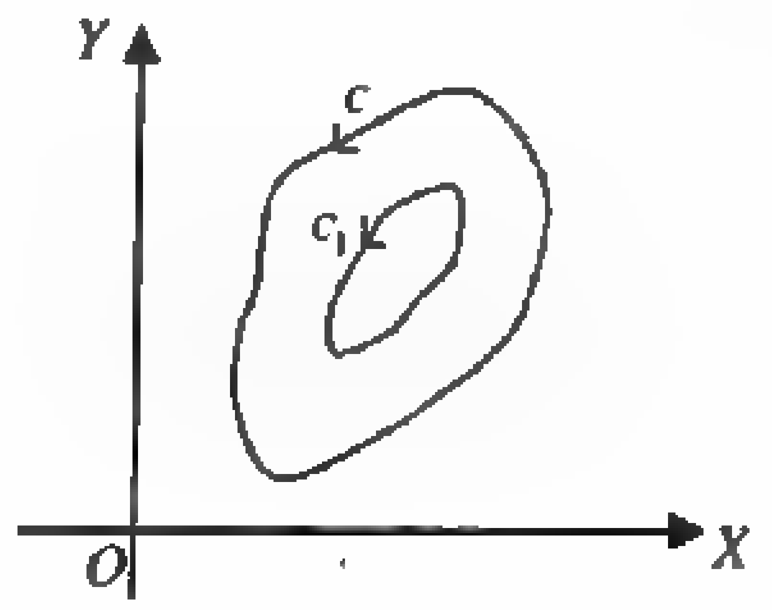
Let f(z) be analytic in a simply-connected region R. Then the following theorems hold.

Theorem 3.4.1.1. *If a and z are any two points in R, then* $\int_a^z f(z)\,dz$ *is independent of the path in R joining a and z.*

Theorem 3.4.1.2. Cauchy's Theorem for Multiple Connected Regions:

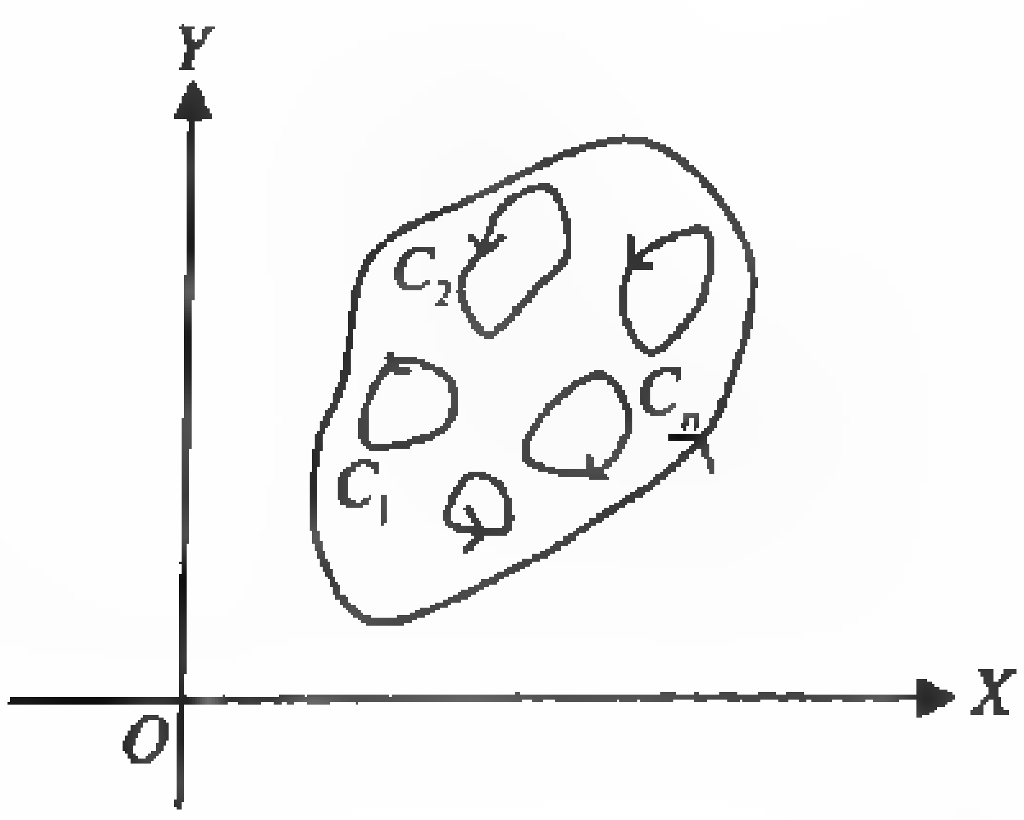
Let f(z) be analytic in a region bounded by two simple closed curves C and C_1 *[where* C_1 *lies inside C as in Fig. below] and on these curves.*

Then $\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$, *where C and* C_1 *are both traversed in the positive sense relative to their interiors.*



Extension: *If C is a closed curve and* $C_1, C_2, C_3, \ldots$, *are the other closed curves which lie inside C, and if a function f(z) is analytic in the region between these curves and is continuous on C, then*

$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz$, *where integral along each curve is taken in the anti – clockwise direction.*



Theorem 3.4.1.3. *If* z_0 *and z are any two points in R and* $G(z) = \int_{z_0}^{z} f(z)\,dz$. *Then G(z) is analytic in R and* $G'(z) = f(z)$.

Theorem 3.4.1.4. *If a and b are any two points in R and $F'(z) = f(z)$, then $\int_a^b f(z)\,dz = F(b) - F(a)$*

This can also be written in the form, $\int_a^b f(z)\,dz = F(z)\,\big|_a^b = F(b) - F(a)$

Example: 3.4.1.4.1. Evaluate $\int_{2i}^{3-i} 2z\,dz$

Solution: $\int_{2i}^{3-i} 2z\,dz = z^2\Big|_{2i}^{3-i} = (3-i)^2 - (2i)^2 = 12 - 6i$ $\quad 9 - 1 + 6i + 4 = 12 - 6i$

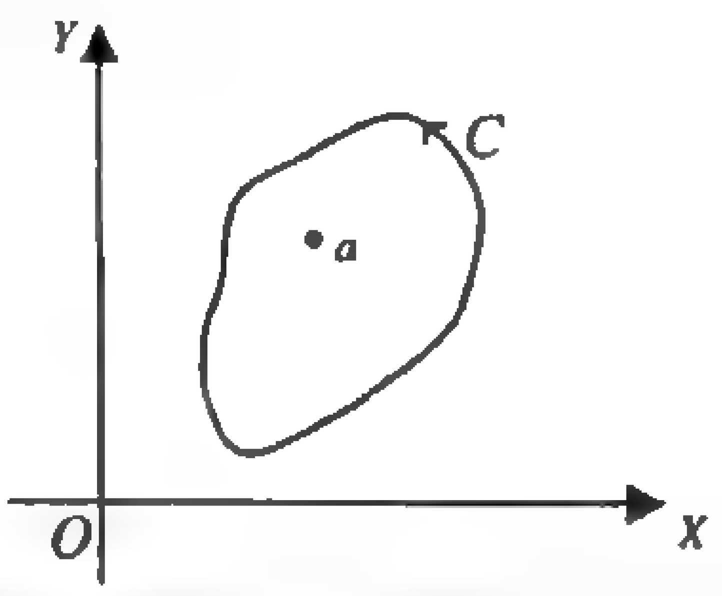
Example: 3.4.1.4.2. *Find* $\int_C (z+2)e^{iz}\,dz$ *along the parabola C defined by* $\pi^2 y = x^2$ *from (0,0) to* $(\pi, 1)$.

Solution: *The point (0,0) and* $(\pi,1)$ *correspond to z=0 and z=π+i. Since* $(z+2)e^{iz}$ *is analytic, Therefore*

$$\int_0^{\pi+i} (z+2)e^{iz}\,dz = \left\{(z+2)\left(\frac{e^{iz}}{i}\right) - (1)(-e^{iz})\right\}\Bigg|_0^{\pi+i} = (\pi+i+2)\left(\frac{e^{i(\pi+i)}}{i}\right) + e^{i(\pi+i)} - \frac{2}{i} - 1$$

$$= -2e^{-1} - 1 + i(2 + \pi e^{-1} + 2e^{-1})$$

3.5. *CAUCHY'S INTEGRAL FORMULAE*



If f(z) is analytic inside and on a simple closed curve C, taken in the positive (anti-clockwise) sense, and a is any point inside C, then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$, ...(1)

Also, the nth derivative of f(z) at z = a is given by $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$, *n=1,2,3, ...* ...(2)

Definition: *If γ is a closed rectifiable curve in C, then for* $a \notin \{\gamma\}$, $n(\gamma{:}a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}\,dz$ *is called the index of γ with respect to the point 'a'. It is also sometimes called the winding number of γ around 'a'. We have*

(i) *If* $\gamma : [0, 1] \to \mathbb{C}$ *is a closed rectifiable curves and* $a \notin \{\gamma\}$, *then* $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ *is an integer.*

(ii) *If γ and σ are closed rectifiable curves having the same initial points, then*
(a) $n(\gamma; a) = -n(-\gamma; a)$ *for every* $a \notin \{\gamma\}$.

(b) $n(\gamma+\sigma;a)=n(\gamma;a)+n(\sigma;a)$ *for every* $a\notin\{\gamma\}\cup\{\sigma\}$.

(iii) *If* γ *is a closed rectifiable curve in* $\mathbb{C}$. *Then* $n(\gamma;a)$ *is constant for* $a\notin\mathbb{C}-\{\gamma\}$.

PRACTICE SET – I

Exercise 1. *The value of integral* $\oint_C (z^2+2\bar{z}^2)\,dz$ *along* $|z|=1$, *where the curve is taken anti-clockwise is*

(a) $4\pi i$ (b) $2\pi i$ (c) 0 (d) 1

Exercise 2. *Evaluate* $\int_C \frac{dz}{z^2+4}$ *along the*

(i) circle $|z|=1$ (ii) circle $|z-i|=2$ (iii) circle $|z|=4$

Exercise 3. $\oint_C \frac{2z^3}{(z-2)^4}dz$ *along* $|z|=3$ *taken anticlockwise is*

(a) $4\pi i$ (b) $2\pi i$ (c) 0 (d) $3\pi i$

Exercise 4. *Evaluate* $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$, *where* $C: |z|=4$.

Exercise 5. *The integral* $\int_{|z|=2\pi} \frac{\sin z}{(z-\pi)^2}$, *(where the curve is taken anticlockwise) equals to*

(a) $-2\pi i$ (b) $2\pi i$ (c) 0 (d) $4\pi i$

3.6. SOME IMPORTANT THEOREMS

The following is a list of some important theorems which are consequences of Cauchy's integral formulae.

1. **Morera's Theorem:** *If* $f(z)$ *is continuous in a simply-connected region R and for every simple closed curve in R,* $\oint_C f(z)dz=0$, *then* $f(z)$ *is analytic within R.*

2. **Cauchy's Inequality:** *If* $f(z)$ *is analytic inside and on a circle C of radius r and centered at* $z=a$, *then* $|f^n(a)|\le \frac{M.n!}{r^n}$, $n=0,1,2,\dots$, *where M is a constant such that* $|f(z)|<M$ *on C, i.e., M is an upper bound of* $|f(z)|$ *on C.*

3. **Liouville's Theorem:** *Suppose that for all z in the entire complex plane, if* $f(z)$ *is analytic bounded, i.e.,* $|f(z)|<M$ *for some constant M. Then* $f(z)$ *must be a constant.*

Note: (i) *An analytic function with constant argument is constant, i.e.,* $\arg f(z)=$ *constant*

$\Rightarrow f(z) = constant$

(ii) If $f(z)=u+iv$ be an analytic function in a domain D, then $f(z)$ is constant in D, if any one of the following conditions hold

(a) $f'(z)$ vanishes identically in D

(b) $Re(f(z))=u=constant$

(c) $Im(f(z))=v=constant$

(d) $|f(z)|=constant$

(e) $\arg f(z)=constant$

4. ***Poisson's Integral formula:*** If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within C, then $f(a)=\frac{1}{2\pi i}\int_C \frac{(R^2-a\bar{a})f(z)dz}{(z-a)(R^2-z\bar{a})}$

From this, we can deduce the Poisson's formula $f(re^{i\theta})=\frac{1}{2\pi i}\int_0^{\pi}\frac{(R^2-r^2)f(\mathrm{Re}^{i\phi})d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$, where $a = re^{i\theta}$ is any point inside the circle.

5. ***Fundamental Theorem of Algebra:***
Every polynomial equation $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has at least one root. Thus, it follows that $P(z) = 0$ has exactly n roots in $\mathbb{C}$.

6. If $g(z)=\overline{f(z)}$, where $f(z)$ is an analytic function, then $g(z)$ will be analytic iff $f(z) = constant$.

7. ***Maximum Modulus Theorem:*** If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs only on boundary of C.

8. ***Minimum Modulus Theorem:*** If $f(z)$ is non-constant analytic function inside and on a simple closed curve C and $f(z) \neq 0$ inside C, then $|f(z)|$ attains its minimum value only on boundary of C.
In other words, if f is analytic in a domain D, $f(z) \neq 0$ on D and a is a point in D, such that $|f(z)|\geq|f(a)|$ holds for all $z\in D$, then f is a constant

9. ***Reflection principle:*** Let $f(z)$ be analytic in a domain Ω containing a segment of the x-axis and be symmetric to that axis. Then $\overline{f(z)} = f(\bar{z}),\ z \in \Omega$

10. ***Parseval's Formula:*** $\sum_{n=0}^{\infty}|a_n|^2 r^{2n} = \frac{1}{2\pi}\int_{t=0}^{2\pi}|f(r^{it})|^2\, dt.$

11. ***Schwarz Lemma:*** If $f(z)$ is analytic in a domain D defined by $|z|<R$ and satisfies the condition $|f(z)|\leq M$ for all $z\in D$ and $f(0)=0$, then $|f(z)|\leq\frac{M}{R}|z|$.

Also, if equality occurs for any z, then $f(z)=\frac{M}{R}ze^{i\alpha}$, where α is a real constant.

or If $D=\{z:|z|<1\}$ and if $f(z)$ is analytic on D with

(a) $|f(z)|\le 1$ for all $z\in D$

(b) $f(0)=0$, then $|f'(0)|\le 1$ and $|f(z)|\le |z|$ for all z in D. In case of equality $f(z)=e^{i\alpha}z$, where α is real constant. Moreover, if $|f'(0)|=1$ or $|f(z)|=|z|$ for some $z\neq 0$, then $f(w)=cw$ for all $w\in D$.

12 **Riemann Mapping Theorem:** A region D is conformally equivalent to $\mathbb{C}$ if there is analytic function $f:D\to\mathbb{C}$ such that f is one one and $f(D)=\mathbb{C}$. If D is simply connected region which is not the whole plane and let $z_0\in D$, then there is a unique analytic function $f:D\to\mathbb{C}$, with the property.

(a) $f(z_0)=0$ and $f'(z_0)>0$ (b) f is one – one (c) $f(D)=\{z:|z|<1\}$

13. **Open Mapping Theorem:** Let D be a region and suppose f is a non- constant analytic function on D. Then, for any open set U in D, f(U) is open.

PRACTICE SET – II

Exercise 1. If $f(z)=\dfrac{e^{-1}}{z^2}$ and $|z| = 1$, then upper bound of $|f'(0)|$ is

(a) 1 (b) e (c) $\dfrac{1}{e}$ (d) none of these

Exercise 2. If D is the open unit disk in $\mathbb{C}$ and $f:\mathbb{C}\to D$ is analytic with $f(10)=\dfrac{1}{2}$, then $f(10+i)$ is

(a) $\dfrac{1+i}{2}$ (b) $\dfrac{1-i}{2}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{i}{2}$

Exercise 3. The roots of the equation $z^4 - z^2 - 2z + 2 = 0$ is/are

(a) 1 (b) -1

(c) $-1 + i$ (d) $-1 - i$

Exercise 4. Let $f(z)=\dfrac{|z|^3}{1+|z|^2}, z\in\mathbb{C}$ and C is the unit circle centered at origin, then

(a) Maximum value of $|f(z)|$ occurs on C

(b) Minimum value of $|f(z)|$ occurs on C

(c) Both maximum and minimum value of $|f(z)|$ occurs on C

(d) Neither minimum nor maximum value of $|f(z)|$ occurs on C

Exercise 5. A polynomial f(z) of degree n has

(a) atmost n roots (b) atleast n roots

(c) exactly n roots (d) less than or equals to n roots.

COMPLEX ANALYSIS

KEY POINTS

- ***Cauchy's theorem :*** *If f(z) is analytic inside and on a closed curve C, then* $\oint_C f(z)dz = 0$

- *If f(z) is an analytic function, then the integration of f(z) is path independent.*

- ***Morera's theorem:*** *If f(z) is continuous function in a simply connected region R such that* $\oint_C f(z)dz = 0$ *around every simple closed curve C in R, then f(z) is analytic in R.*

- ***Cauchy's Integral formulae:*** *If f(z) is analytic inside and on a simple closed curve C and a is any point inside C, then* $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$ *and* $f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$, $n \in \mathbb{N}$

- ***Liouville's theorem :*** *A bounded entire function is constant.*

- ***Maximum modulus theorem:*** *If f(z) is analytic inside and on a simple closed curve C and is non-constant inside C, then |f(z)| assumes its value on C.*

- ***Minimum modulus theorem:*** *If f(z) is analytic inside and on a simple closed curve C and is non-zero inside C, then |f(z)| assumes its minimum value on C.*

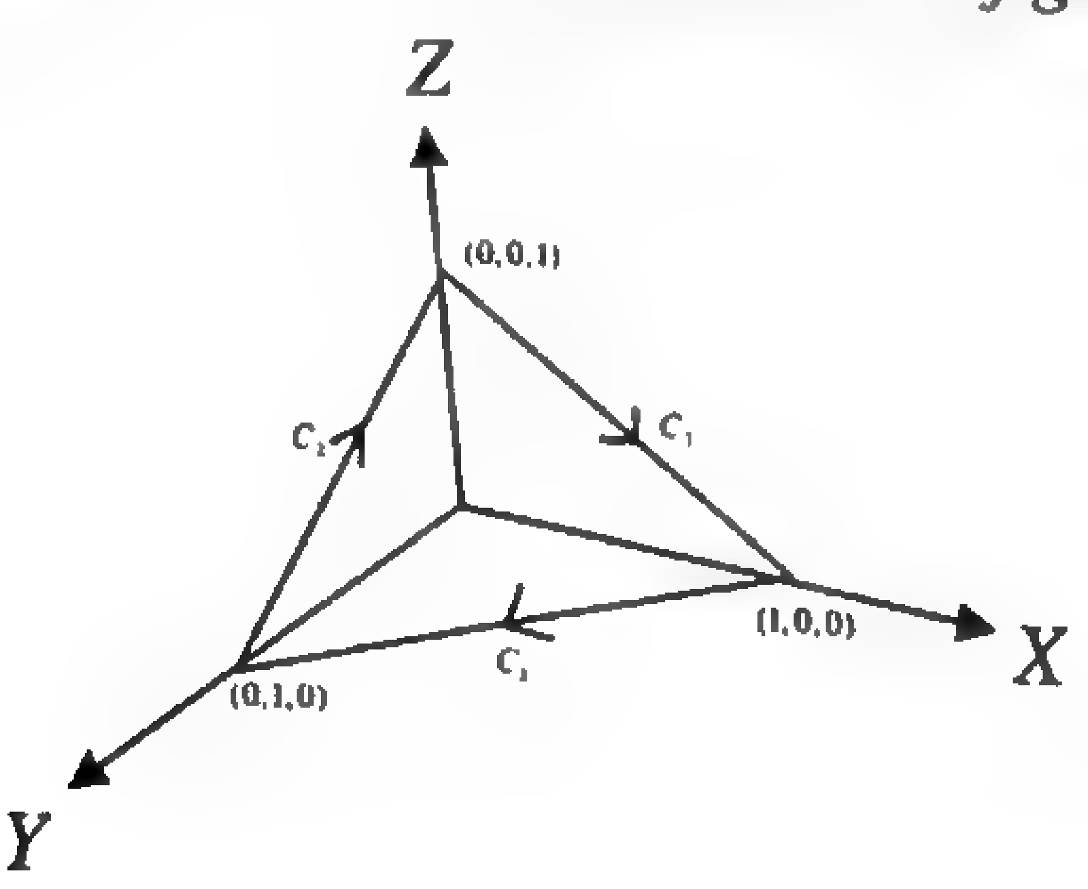
SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *Let C be the boundary of the triangle formed by the points (1, 0, 0), (0, 1, 0), (0, 0, 1). Then the value of the linear integral* $\oint_C -2ydx + (3x - 4y^2)dy + (z^2 + 3y)dz$ *is* ***(GATE-2007)***

(a) 0   *(b) 1*   *(c) 2*   *(d) 4*

Solution: (d) $\oint_C -2y\ dx + (3x - 4y^2)dy + (z^2 + 3y)dz = I_1 + I_2 + I_3$,

where I_i *is integral along* C_i *[C_i's are lines as shown in figure]*



Take $I_1 = \int_0^1 -2t(-dt) + (3(1-t) - 4t^2)dt + 0$

$(\because \text{along } C_1, x = 1-t, y = t, z = 0 \Rightarrow dx = -dt,\ dy = dt,\ dz = 0)$

$$= \int_0^1 (2t+3-3t-4t^2)dt = \frac{-t^2}{2} + 3t - \frac{4t^3}{3}\Big|_0^1 = \frac{-1}{2} + 3 - \frac{4}{3} = \frac{7}{6}$$

$$I_2 = \int_0^1 (0-3(0)+4(1-t)^2 + t^2 + 3 - 3t)dt$$

$(\because \text{along } C_2, x = 0, y = 1-t, z = t \Rightarrow dx = 0,\ dy = -dt,\ dz = dt)$

$$= \int_0^1 (4-8t+4t^2+t^2+3-3t)dt = 7 - \frac{11}{2} + \frac{5}{3} = \frac{19}{6}$$

$$I_3 = \int_0^1 (1-t)^2(-dt)\ (\because \text{along } C_3, x = t, y = 0, z = 1-t \Rightarrow dx = dt, dy = 0,\ dz = -dt)$$

$$= -t + t^2 - \frac{t^3}{3}\Big|_0^1 = \frac{-1}{3}$$

$$\therefore \oint_C -2y\,dx + (3x-4y^2)dy + (z^2+3y)dz = \frac{7}{6} + \left(\frac{-1}{3}\right) + \frac{19}{6} = 4$$

Example 2. *Let f(z) be an analytic function. Then the value of* $\int_0^{2\pi} f(e^{it})\cos(t)dt$ *equals* ***(GATE-2007)***

(a) 0 *(b)* $2\pi f(0)$ *(c)* $2\pi f'(0)$ *(d)* $\pi f'(0)$

Solution: (d) $\int_0^{2\pi} f(e^{it})\cos t\, dt = \int_0^{2\pi} f(e^{it})\frac{e^{it}+e^{-it}}{2}dt$

Let C be the circle $|z| = 1$

Put $z = e^{it},\ dz = ie^{it}dt \Rightarrow dt = \frac{dz}{iz}$

$$\int_0^{2\pi} f(e^{it})\cos t dt = \oint_C f(z)\left(\frac{z+z^{-1}}{2}\right)\frac{dz}{iz} = \frac{1}{2i}\oint_C f(z)\frac{(z^2+1)}{z^2}dz$$

$$= \frac{1}{2i}\oint_C f(z)dz + \frac{1}{2i}\oint_C \frac{f(z)}{z^2}dz = 0 + \frac{1}{2i} \times 2\pi i f'(0) = \pi f'(0)$$

$(\because \oint_C f(z)dz = 0$, *by Cauchy's theorem and* $\int_C \frac{f(z)}{z^2}dz = 2\pi i f'(0)$, *by Cauchy's integral formula)*

Example 3. *Let f(z)* $= 2z^2 - 1$. *Then the maximum value of* $|f(z)|$ *on the unit disc* $D = \{z \in \mathbb{C} : |z| \le 1\}$ *is* ***(GATE-2007)***

(a) 1 *(b) 2* *(c) 3* *(d) 4*

Solution: (c) $|f(z)| = |2z^2 - 1| \le 2|z|^2 + 1 \le 3$ *if* $|z| = 1$ *i.e. z is on the boundary*

By maximum modulus principle, maximum value occurs on boundary.

For $z=i, |2z^2-1|=|-3|=3$

Example 4. *Let T be the closed unit disk and ∂T be the unit circle. Then which one of the following holds for every analytic function* $f: T \to \mathbb{C}$. **(GATE-2008)**

(a) $|f|$ attains its minimum and its maximum on ∂T

(b $|f|$ attains its minimum on ∂T but need not attains its maximum on ∂T

(c) $|f|$ attains its maximum on ∂T but need not attains its minimum on ∂T

(d) $|f|$ need not attains its maximum on ∂T and also it need not attains its minimum on ∂T

Solution: (d) *By maximum modulus principal and minimum modulus principle* $|f|$ *will attain its maximum and minimum value on ∂T, iff f is non-zero and non constant on T.*

Example 5. *Let* $f(z)=\sum_{n=0}^{15} z^n$ *for* $z \in \mathbb{C}$. *If C:* $|z-i|=2$, *then* $\int_C \frac{f(z)dz}{(z-i)^{15}} =$ **(GATE-2009)**

(a) $2\pi i(1+15i)$ (b) $2\pi i(1-15i)$ (c) $4\pi i(1+15i)$ (d) $2\pi i$

Solution: (a) *Note that f(z) is a polynomial of degree 15. So it is an entire function.*

(∴ By Cauchy integral formula) :

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = f^n(z_0)\frac{2\pi i}{n!} \Rightarrow \oint_C \frac{f(z)}{(z-i)^{15}}dz = f^{14}(i)\frac{2\pi i}{14!}$$

As $f(z)=z^{15}+z^{14}+...+z+1$

$f^1(z)=15.z^{14}+14.z^{13}+...+1+0$

$\vdots$

$f^{14}(z)=15.14...2.z+14!=14!(15z+1)$

$$\therefore f^{14}(i)\frac{2\pi i}{14!} = \frac{(15i+1)14!2\pi i}{14!} = 2\pi i(1+15i)$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-i)^{15}}dz = 2\pi i(1+15i)$$

Example 6. *Let u(x,y) be the real part of an entire function f(z) = u(x, y) + iv(x, y) for z=x+ iy∈* $\mathbb{C}$. *If C is the positively oriented boundary of a rectangular region R in* $\mathbb{R}^2$, *then* $\oint_C \left[\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy\right] =$ **(GATE-2009)**

(a) 1 (b) 0 (c) 2π (d) π

Solution: (b) $f(z)=u(x,y)+iv(x,y)$

By Green's theorem, we have $\oint_C \left(\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy\right) = \iint_R \left(-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)dxdy$

$$= -\iint_{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)dx\,dy = 0 \qquad \left[\because f \text{ is analytic} \Rightarrow u \text{ is harmonic, i.e., } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\right]$$

Example 7. *Let* $\int_C\left[\frac{1}{(z-2)^4}-\frac{(a-2)^2}{z}+4\right]dz = 4\pi$, *where the close curve C is the triangle having vertices at i,* $\left(\frac{-1-i}{\sqrt{2}}\right)$ *and* $\left(\frac{1-i}{\sqrt{2}}\right)$, *the integral being taken in anti clockwise direction. Then one value of a is*

(GATE-2012)

(a) $1+i$ — *(b)* $2+i$

(c) $3+i$ — *(d)* $4+i$

Solution: (c) *Since z = 2 lies outside the region.*

$\therefore$ *z = 0 is the only singularity of order one.*

$$4\pi = \int_C\left[\frac{1}{(z-2)^4}-\frac{(a-2)^2}{z}+4\right]dz = -\int_C\frac{(a-2)^2}{z}dz \quad \textit{(using Cauchy's theorem)}$$

Solving with the help of integral formulae, we get, $-2\pi i(a-2)^2 = 4\pi$

$\Rightarrow -i(a-2)^2 = 2 \;\Rightarrow (a-2)^2 = 2i$

$\Rightarrow a = 2\pm\sqrt{2i} = 2\pm(1+i) = 3+i, 1-i \quad (\text{As } \sqrt{2i} = 1+i)$

Example 8. *Let f be an analytic function on* $\overline{D}=\{z\in\mathbb{C}:|z|\le 1\}$. *Assume that* $|f(z)|\le 1$ *for each* $z\in\overline{D}$. *Then, which of the following is NOT a possible value of* $(e^f)''(0)$? *(GATE-2013)*

(a) 2 — *(b)* 6

(c) $\frac{7}{9}e^{\frac{1}{9}}$ — *(d)* $\sqrt{2}+i\sqrt{2}$

Solution: (b) Let $g(z)=e^{f(z)}$

$$g''(0)=\frac{2!}{2\pi i}\oint_C\frac{g(z)}{(z-0)^3}dz \quad \Rightarrow \left|(e^f)''(0)\right| = \left|\frac{2!}{2\pi i}\oint_C\frac{e^{f(z)}}{z^3}dz\right|$$

$$\Rightarrow \left|(e^f)''(0)\right| \le \frac{1}{\pi}\oint_C\frac{\left|e^{f(z)}\right|}{\left|z^3\right|}|dz| \le \frac{1}{\pi}e\times 2\pi = 2e$$

So, option (b) is not possible

Example 9. *Let u be real valued harmonic function on* $\mathbb{C}$. *Let* $g:\mathbb{R}^2\to\mathbb{R}$ *be defined by*

$g(x,y)=\int_0^{2\pi}u(e^{i\theta}(x+iy))\sin\theta\,d\theta$. *Which of the following statements is TRUE?* *(GATE-2013)*

(a) g is a harmonic polynomial — *(b) g is a polynomial but not harmonic*

(c) g is harmonic but not a polynomial — *(d) g is neither harmonic nor a polynomial*

Solution: (a) *Since u is harmonic function on* $\mathbb{C}$

Take $u(x, y) = 1$

Clearly, it is harmonic

$$\therefore g(x,y) = \int_0^{2\pi} 1.\sin\theta\, d\theta = -\cos\theta\Big|_0^{2\pi} = -(\cos 2\pi - \cos 0) = 0$$

$\therefore$ *g(x, y) is harmonic polynomial*

Hence, options (b), (c), (d) are incorrect and option (a) is correct.

Example 10. *The maximum modulus of* e^{z^2} *on the set* $S = \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1, 0 \le \mathrm{Im}(z) \le 1\}$ *is* ***(GATE-2014)***

(a) 2/e *(b) e* *(c) e+1* *(d)* e^2

Solution: (b) *By the maximum modulus theorem the maximum is attained at the boundary.*

Let $z = a + ib$

$\therefore e^{z^2} = e^{a^2 - b^2 + 2abi} \Rightarrow |e^{z^2}| = e^{a^2 - b^2}$

Clearly, $|e^{z^2}|$ *is maximum, when* $a^2 - b^2$ *is maximum and maximum value of* $a^2 - b^2$ *is 1.*

$[\because 0 \le a \le 1,\ 0 \le b \le 1]$

$\therefore$ *Maximum value of* e^{z^2} *is e*

Hence, option (b) is correct.

Example 11. *Let* $\Omega = \{z \in \mathbb{C} : Im(z) > 0\}$ *and let C be a smooth curve lying in* Ω *with initial point* $-1 + 2i$ *and final point* $1 + 2i$*. The value of* $\int_C \frac{1+2z}{1+z} dz$ *is* ***(GATE-2014)***

(a) $4 - \frac{1}{2}\ln 2 + i\frac{\pi}{4}$ (b) $-4 + \frac{1}{2}\ln 2 + i\frac{\pi}{4}$

(c) $4 + \frac{1}{2}\ln 2 - i\frac{\pi}{4}$ (d) $4 - \frac{1}{2}\ln 2 + i\frac{\pi}{2}$

Solution:(a) *Let* $f(z) = \frac{1+2z}{1+z}$ *and let* $z = x + iy,\ y = 2$ *i.e.* $z = x + 2i$

$\Rightarrow$ $dz = dx$*, x varying from -1 to 1.*

$$\int_C \frac{1+2z}{1+z}dz = \int_{-1}^{1} \frac{1+2(x+2i)}{1+x+2i}dx = \int_{-1}^{1}\frac{1+4i+2x}{1+2i+x}dx = \int_{-1}^{1}\frac{2x}{1+2i+x}dx + \int_{-1}^{1}\frac{1+4i}{1+2i+x}dx$$

$$= 2\left[\int_{-1}^{1}\frac{1+2i+x}{1+2i+x}dx - \int_{-1}^{1}\frac{1+2i}{1+2i+x}dx\right] + \int_{-1}^{1}\frac{1+4i}{1+2i+x}dx$$

$$= (1+4i)\,[\ln(1+2i+x)]_{-1}^{1} + 2\,[[x]_{-1}^{1} - (1+2i)\,[\ln(1+2i+x)]_{-1}^{1}]$$

$$= (1+4i)\ln\big(2(1+i)\big) - (1+4i)\ln 2i + 4 - (2+4i)\,[\ln\big(2(1+i)\big) - \ln 2i]$$

$$= ln(2(1+i)) + 4iln(2(1+i)) - ln2i - 4iln2i + 4 - 2ln(2(1+i)) + 2ln2i - 4iln(2(1+i)) + 4iln2i$$
$$= ln2i - ln(2(1+i)) + 4$$
$$= \ln\frac{2i}{2(1+i)} + 4 = \ln\left(\frac{i}{1+i}\right) + 4 = \ln\left[\frac{i(1-i)}{2}\right] + 4 = \ln\left(\frac{1+i}{2}\right) + 4$$
$$= \ln\left(\frac{1}{2}+\frac{i}{2}\right) + 4 = \ln\left(\frac{1}{\sqrt{2}}\right) + i\tan^{-1}(1) + 4 = 4 - \frac{1}{2}\ln 2 + i\frac{\pi}{4}$$

Example 12. *If $a \in \mathbb{C}$ with $|a|<1$, then the value of $\frac{(1-|a|^2)}{\pi}\int_\Gamma \frac{|dz|}{|z+a|^2}$, where Γ is the simple closed curve $|z| = 1$ taken with the positive orientation, is* ___2___ **(GATE-2014)**

Solution: (Ans. 2) *Since $a \in \mathbb{C}$ with $|a| < 1$*

Take, $a = 0$

$\therefore \frac{(1-|a|^2)}{\pi}\int_\Gamma \frac{|dz|}{|z+a|^2}$, *where* $\Gamma : |z| = 1$ *reduces to* $\frac{1}{\pi}\int_{|z|=1} \frac{|dz|}{|z|^2}$

$= \frac{1}{\pi}\int_{|z|=1} |dz| = \frac{1}{\pi}\times 2\pi \quad [\because \int_{|z|=1} |dz| = \text{length of arc } \Gamma : |z| = 1 = 2\pi]$

$= 2$

Hence, answer is 2

Example 13. *Let* $C = \{z \in \mathbb{C} : |z-i| = 2\}$. *Then* $\frac{1}{2\pi}\oint_C \frac{z^2-4}{z^2+4}dz$ *is equal to* ___-2___. **(GATE-2015)**

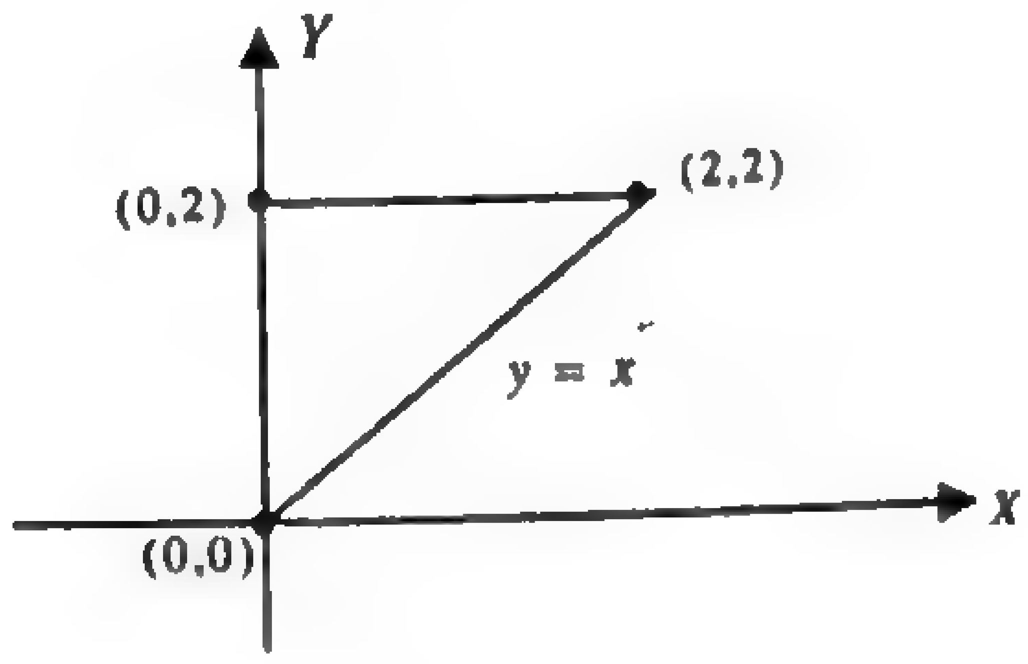
Solution: (Ans. -2) *Given $C = \{z \in \mathbb{C} : |z-i| = 2\}$*

$$\frac{1}{2\pi}\oint_C \frac{z^2-4}{(z-2i)(z+2i)}dz = \frac{1}{2\pi}\oint_C \frac{\left[\frac{(z^2-4)}{(z+2i)}\right]}{(z-2i)}dz = \frac{1}{2\pi}(2\pi i)\left(\frac{-8}{4i}\right) = -2$$

Example 14. *Let γ be the triangular path connecting the points (0,0), (2,2) and (0,2) in the counter-clockwise direction in $\mathbb{R}^2$. Then* $I = \oint_\gamma \sin(x^3)dx + 6xy\,dy$ *is equal to* ___16___ **(GATE-2016)**

Solution: (Ans. 16) *By Green's theorem,* $I = \oint_\gamma \sin(x^3)dx + 6xy\,dy = \int_0^2\int_0^y \left[\frac{\partial}{\partial x}(6xy) - \frac{\partial}{\partial y}(\sin(x^3))\right]dx\,dy$

$$= \int_0^2\int_0^y (6y-0)\,dx\,dy = \int_0^2 6yx\Big|_0^y dy = \int_0^2 6y^2 dy = \frac{6y^3}{3}\Big|_0^2 = 2\times(2)^3 = 16.$$



Example 15. Let $I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)}$, where $C_r = \{z \in \mathbb{C} : |z| = r\}, r > 0$. Then

(CSIR UGC NET DEC-2011)

(a) $I_r = 2\pi i$ if $r \in (2,3)$

(b) $I_r = \frac{1}{2}$ if $r \in (0,1)$

(c) $I_r = -2\pi i$ if $r \in (1,2)$

(d) $I_r = 0$ if $r > 3$

Solution: (d) $I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)}$, where $C_r = \{z \in \mathbb{C} : |z| = r\}$, $r > 0$

Using partial fractions, $\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{-1}{(z-1)} + \frac{1}{2(z-2)}$

$I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)} = \int_{C_r} \left(\frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}\right) dz$

For option (a), $r \in (2,3)$

Let $r = \frac{5}{2} \Rightarrow C_r = \{z \in \mathbb{C} : |z| = \frac{5}{2}\}$

$I_r = \frac{1}{2} \times 2\pi i\, f(0) - 2\pi i\, f(1) + \frac{1}{2} \times 2\pi i\, f(2)$

$\Rightarrow$ option (a) is incorrect

For option (b), $r \in (0,1)$

Let $r = \frac{1}{2} \Rightarrow C_r$

$= \{z \in \mathbb{C} : |z| = \frac{1}{2}\}$

$I_r = \int_{C_r} \frac{\frac{1}{(z-1)(z-2)}}{z} dz = 2\pi i\, f(0) = 2\pi i \left(\frac{1}{(0-1)(0-2)}\right) = \frac{2\pi i}{2} = \pi i$

$\Rightarrow$ option (b) is incorrect

For option (c), $r \in (1,2)$

Let $r = \frac{3}{2} \Rightarrow C_r = \{z \in \mathbb{C} : |z| = \frac{3}{2}\}$

$$I_r = \int_{C_r} \frac{1}{z(z-1)(z-2)} dz = 2\pi i \left[\int_{C_r} \frac{\frac{1}{(z-1)(z-2)}}{z} dz + \int_{C_r} \frac{\frac{1}{(z-2)(z)}}{z-2} dz \right]$$

$$= 2\pi i \left[\lim_{z\to 0} \frac{1}{(z-1)(z-2)} + \lim_{z\to 1} \frac{1}{(z-2)(z)} \right] = 2\pi i \left[\frac{1}{2} - 1\right] = 2\pi i \left(-\frac{1}{2}\right) = -\pi i$$

⇒ option (c) is also incorrect.
As all other options are incorrect
∴ option (d) is correct.

Example 16. *Let $\gamma_k = \{ke^{i\theta} : 0 \le \theta \le 2\pi\}$ for $k = 1, 2, 3$. Which of the following are necessarily correct?*
(CSIR UGC NET DEC-2012)

(a) $\frac{1}{2\pi i}\int_{\gamma_k} \frac{1}{z} dz = 0$ for $k = 1, 2, 3$.
(b) $\frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z} dz = 1$
(c) $\frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{z} dz = 4$
(d) $\frac{1}{2\pi i}\int_{\gamma_3} \frac{1}{z} dz = 3$

Solution: (b,d) *$\gamma_k = \{ke^{i\theta} : 0 \le \theta \le 2\pi\}$ for $k = 1, 2, 3$.*
For options (a) and (b)
$\gamma_1 = \{1e^{i\theta} : 0 \le \theta \le 2\pi\}$ ⇒ $z = 0$ is inside γ_1

So $\frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z} dz = \frac{1}{2\pi i}[2\pi i.1] = 1$

∴ option (b) is correct and (a) is incorrect
For option (c)

Take $\frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{z} dz$, $\gamma_2 = \{2e^{i\theta} : 0 \le \theta \le 2\pi\}$ ⇒ $\frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{z} dz = \frac{1}{2\pi i}[2.2\pi i(1)] = 2$

∴ option (c) is incorrect
For option (d)

$\gamma_3 = \{3e^{i\theta} : 0 \le \theta \le 2\pi\}$

$\frac{1}{2\pi i}\int_{\gamma_3} \frac{1}{z} dz = \frac{1}{2\pi i}[3.2\pi i(1)]$

∴ option (d) is correct.

Example 17. *Let a,b,c be non – collinear points in the complex plane and let Δ denote the closed triangular region of the plane with vertices a,b,c. For $z \in \Delta$, let $h(z) = |z-a| \cdot |z-b| \cdot |z-c|$. The maximum value of the function h* ***(CSIR UGC NET DEC-2013)***

(a) is not attained at any point of Δ
(b) is attained at an interior point of Δ
(c) is attained at the centre of gravity of Δ
(d) is attained at a boundary point of Δ

Solution: (d) *Consider* $g(z)=(z-a)(z-b)(z-c)$, *here g(z) is a holomorphic function*

Now, for the given triangular region by maximum modulus principle, the maximum value of $|g(z)|$ *occurs on the boundary of* Δ, *i.e., maximum value of* $h(z)=|g(z)|=|z-a||z-b||z-c|$ *is attained at the boundary point of* Δ

$\therefore$ *option (d) is correct.*

Example 18. $\int\limits_{|z+1|=2} \frac{z^2}{4-z^2}dz=$ **(CSIR UGC NET DEC-2015)**

(a) 0 (b) $-2\pi i$ (c) $2\pi i$ (d) 1

Solution: (c) *Let* $f(z)=\frac{z^2}{4-z^2}=\frac{z^2}{(2-z)(2+z)}$

z = -2 and z = 2 are singularities of $f(z)$, *but z = -2 lies in the circle* $|z+1|=2$

$$\therefore \int\limits_{|z+1|=2} \frac{z^2}{4-z^2}dz = \int\limits_{|z+1|=2} \frac{\frac{z^2}{2-z}}{z+2}dz = 2\pi i\frac{(-2)^2}{2-(-2)} = 2\pi i$$

$\therefore$ *option (c) is correct*

Example 19. *Let C be the circle* $|z| = 3/2$ *in the complex plane that is oriented in the counter clockwise direction. The value of a for which* $\int\limits_C \left(\frac{z+1}{z^2-3z+2}+\frac{a}{z-1}\right)dz=0$ *is* **(CSIR UGC NET DEC-2016)**

(a) 1 (b) -1 (c) 2 (d) -2

Solution: (c) *Given* $\int\limits_{|z|=\frac{3}{2}} \left(\frac{z+1}{z^2-3z+2}+\frac{a}{z-1}\right)dz=0 \Rightarrow \int\limits_{|z|=\frac{3}{2}} \left(\frac{z+1}{(z-1)(z-2)}+\frac{a}{z-1}\right)dz=0$

Singularities in $|z|=\frac{3}{2}$ *are at z = 1*

$$\Rightarrow \int\limits_{|z|=\frac{3}{2}} \left[\left(\frac{\frac{(z+1)}{(z-2)}}{z-1}\right)+\left(\frac{a}{z-1}\right)\right]dz=0 \Rightarrow \int\limits_{|z|=\frac{3}{2}} \frac{\frac{(z+1)}{(z-2)}}{z-1}dz+\int\frac{a}{z-1}dz=0 \quad ...(1)$$

By Cauchy's integral formula, $\oint\limits_C \frac{f(z)}{z-a}dz=f(a)2\pi i$

$\therefore$ (1) $2\pi i\left[\frac{1+1}{1-2}\right]+a\,2\pi i=0 \Rightarrow 2\pi i[(-2)+a]=0 \Rightarrow a=2$

Hence, option (c) is correct.

ASSIGNMENT - 3.1

NOTE: CHOOSE THE BEST OPTION

1. If $f(z)$ is continuous in a simple connected domain D and if $\oint_C f(z)\,dz = 0$ for every closed contour in D, then
   (a) $f(z)$ is non-analytic in D
   (b) $f(z)$ is analytic in D
   (c) $f(z)$ is constant
   (d) none of these

2. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is
   (a) dependent of path in D
   (b) independent of path in D
   (c) zero
   (d) none of these

3. Let C be the curve of integration, L the length of C and M any constant such that $|f(z)| \le M$ everywhere on C, then the complex line integral is
   (a) $\left|\int_C f(z)dz\right| \le ML$
   (b) $\left|\int_C f(z)dz\right| \le M/L$
   (c) $\left|\int_C f(z)dz\right| \le L/M$
   (d) none of these

4. If f is analytic in a ball $B(a; R)$ and $|f(z)| \le M \; \forall \; z \in B(a; R)$, then
   (a) $|f^{(n)}(a)| \le \dfrac{n!M}{R^n}$
   (b) $|f^{(n)}(a)| \ge \dfrac{n!M}{R^n}$
   (c) $|f^{(n)}(a)| = n!M$
   (d) none of these

5. If γ and σ are closed rectifiable curves having same initial points. If $n(\gamma, a)$ denotes winding number of a, then for every $a \notin \{\gamma\} \cup \{\sigma\}$
   (a) $n(\gamma + \sigma; a) = n(\gamma; a)$
   (b) $n(\gamma + \sigma; a) = n(\sigma; a)$
   (c) $n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a)$
   (d) none of these

6. If γ is a rectifiable curve and f is a continuous function on $\{\gamma\}$, then
   (a) $\left|\int_\gamma f(z)dz\right| \le \int_\gamma |f(z)||dz|$
   (b) $\left|\int_\gamma f(z)dz\right| \ge \int_\gamma |f(z)||dz|$
   (c) $\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)||dz|$
   (d) none of these

7. *If γ and σ are closed rectifiable curves having same initial points, then for every $a \notin \{\gamma\}$*

(a) $n(\gamma; a) = -n(\gamma; a)$

(b) $n(\gamma; a) = n(-\gamma; a)$

(c) $n(\gamma; a) = -n(-\gamma; a)$

(d) none of these

8. *If γ is a closed rectifiable curve in $\mathbb{C}$, then for $a \notin \{\gamma\}$, the index of γ with respect to point a is*

(a) $n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma z\,dz$

(b) $n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}\,dz$

(c) $n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma a\,dz$

(d) none of these

9. The value of $\int_C \frac{(3z^2+7z+1)}{z+1}dz$, where C is $|z|=1/2$ is

(a) $2\pi i$

(b) 0

(c) πi

(d) $\pi i/2$

10. *The integral of $\oint_C (z-z_0)^m\,dz$ (z_0 inside C) is*

(a) 0 for $m = -1$

(b) $2\pi i$ for $m = -1$

(c) 2 for $m = -1$

(d) π for $m = -1$

11. *F is analytic in a disk B(a; R) and γ is closed rectifiable curve in B(a; R), then*

(a) $\int_\gamma F = 0$

(b) $\int_\gamma F \neq 0$

(c) $\int_\gamma F = r$

(d) none of these

12. *If γ is piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous, then*

(a) $\int_a^b f\,d\gamma = \int_a^b f(t)\,\gamma'(t)\,dt$

(b) $\int_a^b f\,d\gamma = \int_a^b f(t)\,\gamma(t)dt$

(c) $\int_a^b f\,d\gamma = \int_a^b f'(t)\,\gamma(t)\,dt$

(d) none of these

13. *If γ is a rectifiable curve and f is continuous function on $\{\gamma\}$, then*

(a) $\int_\gamma f = -\int_\gamma f$

(b) $\int_\gamma f\,dt = -\int_{-\gamma} f\,dt$

(c) $\int_\gamma f\,dt = -\int_\gamma f\,dt$

(d) none of these

14. If γ is a rectifiable curve and f is continuous function on $\{\gamma\}$, then if $c \in \mathbb{C}$

(a) $\int_{\gamma} f(z)dz = \int_{\gamma+c} f(z-c)\,dz$
(b) $\int_{\gamma} f(z)dz = \int_{\gamma} f(z-c)\,dz$
(c) $\int_{\gamma} f(z)dz = \int_{\gamma} f(c)\,dz$
(d) none of these

15. If $\gamma : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then $\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}$ is

(a) an integer
(b) rational number
(c) real number
(d) complex number

16. If $f(z)=u+iv$, then $\int_{c} f(z)\,dz$ is

(a) $\int_C udx - vdy + i\int_C vdx + udy$
(b) $\int_C udx + vdy + i\int_C -udx + vdy$
(c) $\int_C udx - vdy + i\int_C vdx - udy$
(d) $\int_C udx + vdy + i\int_C vdx - udy$

17. For every path between the limits $\int_{-2}^{-2+i}(2+z)^2dz =$

(a) $i/2$
(b) $i/3$
(c) $-i/3$
(d) $-i/2$

18. If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, then

(a) f has a primitive in G
(b) f has no primitive in G
(c) f is constant in G
(d) none of these

19. $\int_L dz$, where L is any rectifiable arc joining the points $z = a$ and $z = b$ is equal to

(a) z
(b) $b - a$
(c) $a - b - z$
(d) $z - a - b$

20. $\int_L |dz|$, where L is any rectifiable arc joining the points $z=a$ and $z=b$ is equal to

(a) $b-a$
(b) $|b-a|$
(c) arc length of L
(d) 0

21. The value of $\int_0^{2+i}(\bar{z})^2dz$ along the line $2y=x$ is

(a) $\frac{5}{3}(2+i)$
(b) $\frac{5}{3}(2-i)$

(c) $(2-i)$ (d) none of these

22. If $f(z)=u+iv$ be continuous in a simply connected region $S\subseteq\mathbb{C}$ and that u and v have continuous partial derivatives and $\oint_C f(z)dz=0$ around every closed curve C in S, then $f(z)$ is

(a) not analytic in S
(b) may or may not be analytic in S
(c) analytic in S
(d) none of the above

23. Let G be a connected open set and $f:G\to\mathbb{C}$ an analytic function. If $a\in G$ such that $f^{(n)}(a)=0, n\geq 0$, then

(a) $f(a)=0$
(b) $f(a)$ = constant
(c) $f(a)=1$
(d) $f(a)=2$

24. If $f(z)$ be an entire function and $u(x,y)=\text{Re}\, f(z)$ is bounded in $\mathbb{C}$, then

(a) $f(z)$ is constant
(b) only $u(x,y)$ is constant
(c) $u(x,y)$ is not constant
(d) none of these

25. The value of $\int_C \frac{(3z+4)}{z(2z+1)}dz$, where C is the circle $|z|=1$ is

(a) $2\pi i$
(b) $3\pi i$
(c) 4
(d) -4

26. The value of the integral $\int_{-\pi}^{\pi}\frac{dt}{4+r^2-4r\cos(\theta-t)}$, $-\pi<\theta\leq\pi$, $0<r<2$, is

(a) 0
(b) $2\pi r^2(\cos\theta-\sin\theta)$
(c) $\frac{4-r^2}{2\pi}$
(d) $\frac{2\pi}{4-r^2}$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

27. If $f(z)$ is analytic in a simple connected domain D, then for every simple closed path C in D which of the following is/are not true?

(a) $\oint_C f(z)\,dz=0$
(b) $\oint_C f(z)\,dz>0$
(c) $\oint_C f(z)\,dz<0$
(d) none of these

28. The complex line integral is generally not

(a) dependent of path
(b) independent of path
(c) independent of end points
(d) none of these

29. *An analytic function is*
(a) *infinitely differentiable*
(b) *finitely differentiable*
(c) *not differentiable*
(d) *continuous*

30. *Which of the following is false?*
(a) *Morera's theorem is exact converse of Cauchy 's theorem*
(b) *Morera's theorem is valid for all functions f(z) in a domain for which* $\int_C f(z)dz = 0$
(c) *Morera's theorem is short converse of Cauchy's theorem*
(d) *All statements are false*

31. *If f(z) is analytic in a domain D, then the false statements are*
(a) $f^{(n)}(z)$ *exist in D*
(b) $f^{(n)}(z)$ *does not exist in D*
(c) $f^{(n)}(z) = 0$ *for all n in D*
(d) $f^{(n)}(z) \neq 0$ *for all n in D*

32. *Which of the following statements are true?*
(a) *If f is bounded function, then* $|f(z)| \leq M$ *for all* $z \in \mathbb{C}$.
(b) *If f is bounded function, then* $|f(z)| \geq M$ *for all* $z \in \mathbb{C}$.
(c) *If f is bounded entire function, then f is constant*
(d) *If f is bounded entire function, then f is not constant.*

33. *If f(z) is analytic within and on a closed curve C and if* α *is any point within C, then*
(a) $\int_C \frac{f(z)}{(z-\alpha)} dz = 2\pi i f(\alpha)$
(b) $\int_C \frac{f(z)}{(z-\alpha)} dz = 2\pi i f'(\alpha)$
(c) $\int_C \frac{f(z)}{(z-\alpha)} dz = \pi i f(\alpha)$
(d) $\int_C \frac{f(z)}{(z-\alpha)^2} dz = 2\pi i f'(\alpha)$

34. *If G is a region and* $f: G \to \mathbb{C}$ *is continuous function such that* $\int_T f = 0$ *for every triangular path (closed curve)* $T \subset G$, *then*
(a) *f is analytic in G*
(b) *f is not analytic in G*
(c) *Both (a) and (b)*
(d) *None of these*

35. *Let* $I = \int_\gamma \frac{dz}{z-a}$, *where* γ *represents the circle |z-a|=r,then*
(a) *the value of I is* $2\pi i$
(b) *the value of I is* π
(c) *the value of I is zero*
(d) *the value of I is one*

36. *If f(z) is analytic in a simply connected domain D, then which of the following is/are false?*
(a) *Indefinite integral exist*
(b) *Indefinite integral does not exist*
(c) *Integral of f(z) is dependent of path in D*
(d) *Integral of f(z) is independent of path in D*

37. *A path in some region is not*
(a) continuous function
(b) discontinuous function
(c) necessarily differentiable function
(d) necessarily closed

38. *Let C be the the square with vertices at (0,0) (1,0) (1,1) and (0,1), then*
(a) Square is closed and $|z|$ is analytic so by Cauchy's theorem $\oint_C |z|dz = 0$
(b) $\oint_C |z|^2 dz$ can't be determined because $|z|$ is not analytic and square is closed.
(c) $\oint_C |z|^2 dz = a + ib$, where a = -1, b=1
(d) None of these

39. *A path is said to be smooth path if*
(a) it is continuous function
(b) it is continuous and differentiable function
(c) not differentiable function
(d) none of these

40. *If $f : G \to \mathbb{C}$ is analytic and $\overline{B}(a;r) \subset G$, then*
(a) $f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw$, where $\gamma(t) = a + re^{it}$ and $0 \le t \le 2\pi$
(b) $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a} dw$, where $\gamma(t) = a + re^{it}$ and $t \in [0, 2\pi]$
(c) $f^{(n)}(a)$ does not exists
(d) $f^{(n)}(a)$ exists

41. *Let $f(z) = \oint_\gamma \phi(z)dz$, where $\phi(z) = \frac{e^{-z}}{z+1}$ and $\gamma : |z| = \frac{1}{2}$, then*
(a) $\phi(z)$ is analytic outside $|z| = \frac{1}{2}$
(b) $\phi(z)$ is analytic on $|z| = \frac{1}{2}$
(c) $\phi(z)$ is analytic in $|z| < \frac{1}{2}$, $\forall\, z \in \mathbb{C}$
(d) $f(z) = 0$

42. *Which of the following is not the value of the integral $\oint_C \frac{e^z}{z-2} dz$, where C : |z| = 3*
(a) $2\pi i$
(b) e^2
(c) $2\pi i e$
(d) $2\pi i e^2$

ASSIGNMENT - 3.2

NOTE: CHOOSE THE BEST OPTION

1. The value of $\oint_C \frac{z^3-6}{2z-i}dz$, where C is $|z| = 1$ is

(a) $\pi/8$ (b) $6\pi i$
(c) $(\pi/8) - 6\pi i$ (d) $-6\pi i - (\pi/8)$

2. $\oint_C \frac{\tan z}{z^2-1}dz$ has the value around the circle $C : |z| = 3/2$ (positive oriented),

(a) $\tan 1$ (b) $2\pi i \tan 1$
(c) $\frac{\tan 1}{2\pi i}$ (d) none of these

3. The value of the line integral $\oint_C \frac{-ydx+xdy}{x^2+y^2}$, where C is the unit circle centered at O, equals

(a) 2π (b) -2π
(c) 0 (d) none of these

4. $\int e^{-2z} dz$ over $|z| = 1$ is

(a) $-\frac{1}{2}e^{-2z} + C$ (b) $e^{-2z} + C$
(c) 0 (d) πi

5. If C is the curve $y=x^3-3x^2+4x-1$, joining the two points (1,1) and (2,3). The value of $\int_C (12z^2-4iz)dz$ is

(a) $-156+38i$ (b) $156-38i$
(c) $38+156i$ (d) $38-156i$

6. The value of $\frac{2!}{2\pi i}\int_{|z|=3} \frac{z^2+3z+4}{(z-1)^3}dz$ is

(a) 2 (b) 0
(c) πi (d) none of these

7. Let f(z) be defined in $\{(x, y) : |x| \le 4, |y| \le 3\}$ If f(z) is analytic in D and satisfies $|f(z)| \le 1$ on ∂D then $|f'(0)| \le$

(a) zero (b) $14/9\pi$
(c) $-i\pi/4$ (d) none of these

8. *Suppose that a function f is continuous in a domain D then among the following statements*
   (I) *f has primitive in D*
   (II) *The integral of f (z) along any path lying in D between any two fixed points in D is independent of path*
   (III) *The integral of f (z) along every closed contour in D is zero*
   (a) *I implies III but not II*
   (b) *II implies I but II not implied by I*
   (c) *I Implies II & III but not implied by either II or III*
   (d) *All the statement are equivalent*

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

9. *If* $f(z)=\dfrac{z^2+5z+6}{z-2}$ *and the path of integration is a circle with origin as center and radius r, then the Cauchy's theorem is not applicable, whenever r equals*
   (a) 1 (b) 2
   (c) 3 (d) 4

10. *If* $f(a)=\int_{\gamma}\dfrac{3z^2+7z+1}{z-a}\,dz$, *where* $\gamma:|z|=2$, *then*
   (a) $f(0)=2\pi i$ (b) $f'(0)=14\pi i$
   (c) $f''(-1)=16\pi i$ (d) $f''(-2)=12\pi i$

11. *If* $\int_{\gamma}\dfrac{z+4}{z^2+2z+5}\,dz=0$, *then*
   (a) $\gamma:|z|=1$ (b) $\gamma:|z+1|=1$
   (c) $\gamma:|z|=3$ (d) $\gamma:|z+1|=\dfrac{1}{2}$

12. *Which of the following is/are correct?*
   (a) *If f(z) is analytic at all points in a region* $S\in E_2$ *and C is a curve lying in the above region S, then the complex function f(z) is definitely integrable along the curve C.*
   (b) $\int_{[a,b]} z\,dz=\dfrac{1}{2}(a^2-b^2)$
   (c) $\int_C \bar{z}\,dz$ *from z=0 to z=4+2i along the curve C given by* $z=t^2+it$ *is (10-8i/3)*
   (d) *None of the above*

13. *Let f be an entire function. If f satisfies the following two equations f(z + 1) = f(z) and f(z + i) = f(z) for every z in* $\mathbb{C}$, *then which of the following is false ?*
   (a) $f'(z)=f(z)$ (b) $f(z)\to\mathbb{R}\ \forall z$
   (c) $f=$ *constant* (d) *f is a non-constant polynomial*

ASSIGNMENT - 3.3

NOTE: CHOOSE THE BEST OPTION

1. Let $C: z = 2e^{i\theta}, 0 \le \theta \le 2\pi$. Then $\left|\int_C \frac{e^z}{z^2+1} dz\right| \le$

(a) zero
(b) $\pi e^2/3$
(c) $2\pi e^2/3$
(d) $4\pi e^2/3$

2. Let C: $|z - i| = 2$ be in positive sense. Then $\int_C \frac{dz}{(z^2+4)^3} =$

(a) zero
(b) $3\pi/256$
(c) $-3i\pi/256$
(d) none of these

3. Let C: $|z| = 4$. Then $\int_C \frac{z^2 e^z \sin z}{z-8} dz =$

(a) zero
(b) $i\pi/2$
(c) $-i\pi/4$
(d) none of these

4. If C is an ellipse with positive orientation with the centre at the origin, then $\int_C \frac{z^3+3}{z} dz =$

(a) zero
(b) $i\pi/2$
(c) $-i\pi/4$
(d) $6\ i\pi$

5. Let C: $|z - 1| = 2$, taken in positive sense. Then $\int \frac{z^3 \sin z}{(z-1)^3} dz =$

(a) zero
(b) $i\pi/2$
(c) $-i\pi/4$
(d) none of these

6. The integral along $|z| = 2$ (positively oriented) $\int \frac{\cos z}{z(z^2+9)} dz$ has the value

(a) zero
(b) $i\pi/2$
(c) $-i\pi/4$
(d) none of these

7. The value of the integral $\int_C z^2 dz$, where C is the line segment joining (0, 0) and (1, 1), is

(a) zero
(b) $(1+i)^3/3$
(c) $(1-i)/3$
(d) none of these

8. If C is the boundary of the square $\{-1 \le x \le 1, -1 \le y \le 1\}$ described in the anticlockwise direction, then the value of the integral $\int_C \frac{z^2}{4z-1-i} dz$ is

(a) zero
(b) $-\pi/16$
(c) $-i\pi/4$
(d) $i\pi/2$

9. The value of the integral $\int_{|z|=3} \frac{\cos z}{z(z^2+14)} dz$ is

(a) zero
(b) $2\pi i/7$
(c) $i\pi/7$
(d) none of these

10. The value of the integral $\int_{|z+\pi|=2} \frac{\sin z}{e^z - 1} dz$

(a) zero
(b) $2\pi i$
(c) 4π
(d) none of these

11. If $F(z, \bar{z})$ be continuous and have continuous partial derivatives in a region $S \in E_2$ and on its boundary C, then $\oint_C F(z,\bar{z})dz = 2i\iint_S G(z)dA$, where dA represents the element of area dx dy. Then G(z)=

(a) $\frac{\partial F}{\partial z}$
(b) $\frac{\partial F}{\partial \bar{z}}$
(c) $\frac{\partial^2 F}{\partial z^2}$
(d) None of these

12. Let C be the ellipse defined by $4x^2 + 9y^2 = 36$. Then the value of $\int_C \{(y^2x - 2y + 3)dx + (x^2y + 3x + 7)dy\}$ is

(a) 6π
(b) 30π
(c) 60π
(d) 210π

13. Let γ be the curve : $r = 2 + 4\cos\theta$, $(0 \le \theta \le 2\pi)$. If $I_1 = \int_\gamma \frac{dz}{z-1}$ and $I_2 = \int_\gamma \frac{dz}{z-3}$, then

(a) $i_1 = 2i_2$
(b) $i_1 = i_2$
(c) $2i_1 = i_2$
(d) $i_1 = 0, i_2 \ne 0$

14. Let f(z) be defined on the domain $E: |z-2i| < 3$ and on its boundary ∂E. Then which of the following statements is always true?

(a) If f(z) is analytic on E and $f(z) \ne 0$ for any z in E, then $|f|$ attains its maximum on ∂E
(b) If f(z) is analytic on $E \cup \partial E$, then $|f|$ attains its minimum on ∂E
(c) If f(z) is analytic on E and continuous on $E \cup \partial E$, then $|f|$ attains its maximum and minimum on ∂E
(d) If f(z) is analytic on $E \cup \partial E$ and $f(z) \ne 0$ for any z in $E \cup \partial E$, then $|f|$ attains its minimum on ∂E

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

15. *The curve represented by the function* $\cos t + i \sin t$, $-\pi \le t \le \pi$ *is not*

(a) straight line
(b) parabola
(c) circle
(d) ellipse

16. $\oint_C \frac{(x+1)dy - ydx}{(x+1)^2 + y^2}$ *where C is* $|z+1| = 1$ *is not*

(a) 0
(b) π
(c) 2π
(d) 3π

17. *If C is the boundary of the square.* $\{-1 \le x \le 1, -1 \le y \le 1\}$ *described in anticlockwise direction, Then the value of the integral* $\int_C \frac{z^2}{[z-(1+i)/4]^5} dz$ *cannot be*

(a) zero
(b) $i\pi/2$
(c) $-i\pi/4$
(d) $2\pi i$

18. *Let* $A = \int_C \frac{e^{2z} dz}{(z-1)(z-2)}$, *where C is the circle* $|z|=3$ *and* $B = \int_C \frac{\cos \pi z}{z^2-1} dz$ *around a rectangle with vertices* $2 \pm i, -2 \pm i$. *Then*

(a) $A = 2\pi i(e^4 - e^2)$
(b) $B = 0$
(c) $A = 0$
(d) $B = 2\pi i(e^4 - e^2)$

ANSWERS TO EXERCISES

(PRACTICE SET – I)

Exercise 1: *(c)*

Exercise 2: *(i) 0* *(ii)* $\frac{\pi}{2}$ *(iii) 0*

Exercise 3: *(a)*

Exercise 4: $\frac{i}{\pi}$

Exercise 5: *(a)*

(PRACTICE SET – II)

Exercise 1: *(c)*
Exercise 2: *(c)*
Exercise 3: *(a,b,c,d)*
Exercise 4: *(d)*
Exercise 5: *(c,d)*

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 3.1

1. (b)	2. (b)	3. (a)	4. (a)	5. (c)	6. (a)	7. (c)
8. (b)	9. (b)	10. (b)	11. (a)	12. (a)	13. (b)	14. (a)
15. (a)	16. (a)	17. (c)	18. (a)	19. (b)	20. (c)	21. (b)
22. (c)	23. (a)	24. (a)	25. (b)	26. (d)		
27. (b,c,d)	28. (b,c,d)	29. (a,d)	30. (a,b,d)	31. (b,c,d)	32. (a,c)	33. (a,d)
34. (a)	35. (a)	36. (b,c)	37. (b,c,d)	38. (c)	39. (b)	40. (a,b,d)
41. (b,c,d)	42. (a,b,c)					

ASSIGNMENT - 3.2

1. (c)	2. (b)	3. (a)	4. (c)	5. (a)	6. (a)	7. (b)
8. (d)						
9. (b,c,d)	10. (a,b)	11. (a,b,d)	12. (a,c)	13. (a,b,d)		

ASSIGNMENT - 3.3

1. (d)	2. (b)	3. (a)	4. (d)	5. (d)	6. (d)	7. (b)
8. (b)	9. (c)	10. (a)	11. (b)	12. (b)	13. (a)	14. (d)
15. (a, b)	16. (a,b,d)	17. (b,c,d)	18. (a,b)			

CHAPTER - 4
SEQUENCE AND SERIES

INTRODUCTION

We have studied sequence and series of functions in real analysis. A sequence is a function whose domain is set of natural numbers. If co-domain is set of real numbers, then it is called sequence of real numbers and if co-domain is set of complex numbers, then it is called sequence of complex numbers. Series have same meaning in complex analysis, i.e., it is infinite sum of the sequence. In this chapter, we will discuss about sequence and some particular types of series named Power Series, Laurent's Series and Taylor's Series.

4.1. SEQUENCE

4.1.1. Sequence of Numbers:

*A **sequence** is a function of a positive integral variable, denoted by $f(n)$ or $\{u_n\}$, where $n = 1, 2, 3,...$ formed according to a definite rule. Each number in the sequence is called a term and u_n is called the nth term. The sequence is called finite or infinite according as there are finite number of terms or not. Unless otherwise specified, we shall consider infinite sequence only.*

For example:

(i) *The set of numbers $i, i^2, i^3,..., i^{100}$ is a finite sequence; the nth term is given by $u_n = i^n$, $n = 1, 2, ..., 100$.*

(ii) *The set of numbers $1+i, \frac{(1+i)^2}{2!}, \frac{(1+i)^3}{3!},...$ is an infinite sequence; the nth term is given by*

$$u_n=\frac{(1+i)^n}{n!},\ n= 1, 2, 3, ...$$

Convergence of a sequence:

*A number l is called the **limit** of an infinite sequence $u_1, u_2, u_3,...$ if for any positive number $\in$, we can find a positive number N depending upon $\in$ such that $|u_n - l| < \in$ for all $n \geq N$. In such case we write $\lim_{n\to\infty} u_n = l$. If the limit of a sequence exists finitely, then the sequence is called **convergent**, otherwise it is called **divergent**.*

Note: *A more intuitive way of expressing this concept of limit is to say that a sequence $u_1, u_2, u_3,...$ has a limit l if the successive terms get "closer and closer" to l.*

Important theorems on sequences:

Theorem 4.1.1. *If a sequence has a limit, then it is unique [i.e. it is the only one].*

Theorem 4.1.2. *Let $u_n = a_n + ib_n$, $n = 1, 2, 3, ...$, where a_n and b_n are real. Then a necessary and sufficient condition that $\{u_n\}$ converges is that $\{a_n\}$ and $\{b_n\}$ converges.*

Theorem 4.1.3. *A necessary and sufficient condition that $\{u_n\}$ converges is that for any given $\in > 0$, there exists a positive integer N such that $|u_p - u_q| < \in$ for all p, q > N. This result, is known as* ***Cauchy's Convergence Criterion*** *and the sequence $\{u_n\}$ is known as Cauchy Sequence. Every convergent sequence is a Cauchy Sequence.*

Results on limit of sequences:

If $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$, then

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = A + B$

(ii) $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = A - B$

(iii) $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right) = AB$

(iv) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{A}{B}$, if $B \neq 0$.

4.1.2. Sequence of functions:

Let $\{u_n(z)\}$, be a sequence of functions of z defined and single-valued in some region of the z plane. U(z) is called the limit of $u_n(z)$ as $n \to \infty$, if given any positive number $\in$ there exists a positive integer N [depending in general on both $\in$ and z] such that $|u_n(z) - U(z)| < \in$ for all n > N. In such case, the sequence $\{u_n(z)\}$ converges or is convergent to U(z).

Uniform Convergence:

A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S, if for every $\in>0$, there exists a positive integer N (depending on $\in$ only) such that $|f_n(z)-f(z)| < \in\ \forall\ n\geq N,\ z\in S$.

4.2. INFINITE SERIES

4.2.1. Series of numbers:

Let $u_1, u_2, u_3, \ldots$ be a given sequence. Form a new sequence $S_1, S_2, S_3, \ldots$ defined by

$$S_1 = u_1,$$
$$S_2 = u_1 + u_2,$$
$$S_3 = u_1 + u_2 + u_3,$$
$$\vdots$$
$$S_n = u_1 + u_2 + \ldots + u_n$$

Here, S_n is called the nth partial sum of the first n terms of the sequence $\{u_n\}$.

$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \dots + u_n + \dots$ *is called an* ***infinite series****. If* $\lim_{n\to\infty} u_n = u$ *exists finitely, the series is called* ***convergent*** *and u is its sum; otherwise the series is called* ***divergent****. Thus, a series is said to converge iff the corresponding sequence of partial sums is convergent.*

Result: *A necessary condition that a series* $\sum_{n=1}^{\infty} u_n$ *is convergent is* $\lim_{n\to\infty} u_n=0$ *but this is not sufficient.*

4.2.2. Series of functions:

From the sequence of functions $\{u_n(z)\}$ *let us form a new sequence* $\{S_n(z)\}$ *defined by*

$$S_1(z) = u_1(z)$$
$$S_2(z) = u_1(z) + u_2(z)$$
$$\vdots$$
$$S_n(z) = u_1(z) + u_2(z) + \dots + u_n(z),$$

where $S_n(z)$*, called the nth partial sum, is the sum of the first n terms of the sequence* $\{u_n(z)\}$*.*

The sequence $S_1(z)$, $S_2(z)$, ... *or* $\{S_n(z)\}$ *is symbolized by* $u_1(z)+u_2(z)+\dots=\sum_{n=1}^{\infty} u_n(z)$*, is called an infinite series. If* $\lim_{n\to\infty} S_n(z) = S(z)$*, the series is called convergent and S(z) is its sum; otherwise the series is called divergent.*

Cauchy's General Principle of convergence for a series:

The necessary and sufficient condition for the series $\sum_{n=0}^{\infty} u_n z^n$ *to be convergent is that to a given positive number,* $\in$ *there corresponds a positive integer m such that*

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \in \quad \forall\, n \geq m, p \geq 0.$$

Absolute Convergence:

A series $\sum_{n=1}^{\infty} u_n(z)$ *is called absolutely convergent if* $\sum_{n=1}^{\infty} |u_n(z)|$*, converges. If* $\sum_{n=1}^{\infty} u_n(z)$ *converges but* $\sum_{n=1}^{\infty} |u_n(z)|$ *does not converge, then* $\sum_{n=1}^{\infty} u_n(z)$ *is said to be conditionally convergent. Every absolutely convergent series is convergent.*

Uniform Convergence:

Let $\{f_n(z)\}$ *be a sequence of complex valued functions defined on a set S in* $\mathbb{C}$*. Then series* $\sum_{n=1}^{\infty} f_n(z)$ *converges* ***uniformly*** *on the set S if sequence of partial sums* $\{S_n(z)\}$ *converges uniformly on S.*

Theorems on Convergence of series:

A finite number of terms can be added or deleted to a series without altering the convergence or divergence of the series.

Theorem 4.2.1. *Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence. However, in case of convergent series, its sum will change.*

Theorem 4.2.2. *A necessary and sufficient condition that* $\sum_{n=1}^{\infty}(a_n + ib_n)$ *converges (where* a_n *and* b_n *are real), is that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges.*

Theorem 4.2.3. *If* $\sum_{n=1}^{\infty} |u_n|$ *converges, then* $\sum_{n=1}^{\infty} u_n$ *converges, but the converse may not be true.*

Theorem 4.2.4. *Let* $\sum_{n=1}^{\infty} u_n$ *be an absolutely convergent series having sum S. Then every rearrangement of* $\sum_{n=1}^{\infty} u_n$ *also converges absolutely and has the sum S. Also the sum, difference and product of absolutely convergent series is absolutely convergent. These are not so far conditionally convergent series.*

Theorem 4.2.5. (Comparison tests):

(a) If $\sum_{n=1}^{\infty} |v_n|$ *converges and* $|u_n| \le |v_n|$, *then* $\sum_{n=1}^{\infty} u_n$ *converges absolutely.*

(b) If $\sum_{n=1}^{\infty} |v_n|$ *diverges and* $|u_n| \ge |v_n|$, *then* $\sum_{n=1}^{\infty} |u_n|$ *diverges but* $\sum_{n=1}^{\infty} u_n$ *may or may not converge.*

Theorem 4.2.6. (Ratio test):

$\sum_{n=1}^{\infty} u_n$ *is absolutely convergent if* $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l < 1$ *and diverges if* $l > 1$. *If* $l = 1$, *the test fails.*

Theorem 4.2.7. (Dirichlet test):

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series of complex numbers such that*

(i) the partial sums of $\sum_{n=1}^{\infty} a_n$ *are bounded*

(ii) $\{b_n\}$ *is a monotonically decreasing sequence*

(iii) $b_n \to 0$ *as* $n \to \infty$

Then $\sum_{n=1}^{\infty} a_n b_n$ *converges*

Theorem 4.2.8. (Root test):

$\sum_{n=1}^{\infty} u_n$ *is absolutely convergent if* $\lim_{n\to\infty} \sqrt[n]{|u_n|} = l < 1$ *and diverges if* $l > 1$. *If* $l = 1$, *the test fails.*

Theorem 4.2.9. (Integral test):

If $f(x) \geq 0$ *for* $x \geq a$, *then* $\sum f(n)$ *converges or diverges according as* $\lim_{M\to\infty} \int_a^M f(x)\,dx$ *converges or diverges.*

Theorem 4.2.10. (Raabe's test):

If $\lim_{n\to\infty} n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right) = l$, *then* $\sum_{n=1}^{\infty} u_n$ *absolutely converges if* $l > 1$ *and diverges or converges conditionally if* $l < 1$. *If* $l = 1$, *the test fails.*

Theorem 4.2.11. (Alternating series test):

Let $\sum_{n=1}^{\infty} a_n$ *be a series of complex number such that*

1. $a_n \geq 0 \ \forall \ n \in \mathbb{N}$ *2.* $a_{n+1} \leq a_n \ \ \forall \ n \in \mathbb{N}$ *3.* $\lim_{n\to\infty} a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 \ldots$ *is convergent.*

Theorem 4.2.12. (Gauss's test):

If $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^2}$, *where* $|c_n| < M$ *for all* $n > N$, *then* $\sum u_n$ *converges absolutely if* $l > 1$ *and diverges or converges conditionally if* $l \leq 1$.

Theorems on Uniform Convergence:

Theorem 4.2.13. (Weierstrass M-test):

Let $\sum_{n=1}^{\infty} M_n$ *be a convergent series of positive real numbers such that* $|u_n(z)| \leq M_n$ *for all* $n \in \mathbb{N}$ *and for all Z in a region R. Then the series* $\sum_{n=1}^{\infty} u_n(z)$ *converges absolutely and uniformly in R.*

Theorem 4.2.14. *The sum of a uniformly convergent series of continuous functions is continuous, i.e. if $u_n(z)$ is continuous in R and $S(z) = \sum u_n(z)$ is uniformly convergent in R and C is a curve in R, then* $\int_C S(z)\,dz$ $= \int_C u_1(z)\,dz + \int_C u_2(z)\,dz + \ldots$ *or* $\int_C \left(\sum u_n(z)\right)dz = \sum\left(\int_C u_n(z)\,dz\right)$

In other words, a uniformly convergent series of continuous functions can be integrated term by term.

Theorem 4.2.15. *Suppose that*

1. $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly to f(z) in a region R*

2. *Each $f_n(z)$ is analytic in R.*
   Then the sum function f(z) is analytic in the region R. Moreover, the series can be differentiated term by term any number of times, i.e., $\sum_{n=1}^{\infty} f_n^{(m)}(z) = f^{(m)}(z), z \in R,\ m = 1,2,\ldots$ *and each differentiated series is uniformly convergent in R.*

4.3 POWER SERIES

A series of the form $a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *...(1), is called a power series about point z_0, where z_0, a_0, $a_1, a_2, \ldots, a_n$ are complex constants and z_0 is the centre of the series; $a_0, a_1, a_2, \ldots, a_n$ are known as the coefficients of the series. In particular, the series* $\sum_{n=0}^{\infty} a_n z^n$ *is a power series about the point z_0.*

4.3.1. Cauchy-Hadamard Theorem:

For a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ *...(1), there exists a unique real number R ($0 \le R \le \infty$) such that it*

1) *converges absolutely for $|z-a| < R$*

2) *converges uniformly for $|z-a| \le \rho$ ($0 \le \rho < R$)*

3) *diverges for $|z-a| > R$*

R is called the radius of convergence of the power series (1) and the circle $|z|=R$ which also includes in its interior $|z|<R$ all the values of z for which the power series (1) converges is called the circle of convergence of the series.

The radius of convergence is given by the relation : $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

4.3.2. Region of Convergence:

The power series $\sum a_n z^n$, *either*

(i) *Converges for all values of z*

(ii) *Converges only for $z=z_0$*

(iii) *Converges for z in some region in the complex plane.*

Theorems on Power Series:

Theorem 4.3.1.

(i) *A power series converges uniformly and absolutely in any region which lies entirely inside its circle of convergence.*

(ii) *A power series can be differentiated term by term in any region which lies entirely inside its circle of convergence. Thus if $f(z)=\sum_{n=0}^{\infty} a_n z^n$ converges in $|z|<R$, then the derivatives of all orders of $f(z)$ exist in $|z|<R$ and $f^{(m)}(z)=\sum_{n=m}^{\infty} n(n-1)(n-2)....(n-m+1)a_n z^{n-m}$; $m=0,1,2.....$*

(iii) *A power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ can be integrated term by term along any curve C which lies entirely inside its circle of convergence, i.e., $\int_c f(z)dz=\sum_{n=0}^{\infty} a_n \int_c z^n dz$*

(iv) *The sum of a power series is continuous in any region which lies entirely inside its circle of convergence.*

Theorem 4.3.2. (Abel's theorem):

Let $\sum a_n z^n$ have radius of convergence R and suppose that z_0 is a point on the circle of convergence such that $\sum a_n z_0^n$ converges. Then $\lim\limits_{z\to z_0} \sum a_n z^n = \sum a_n z_0^n$.

Theorem 4.3.3. *If $\sum a_n z^n$ converges to zero for all z such that $|z| < R$, where $R > 0$, then $a_n = 0$. Equivalently, if $\sum a_n z^n = \sum b_n z^n$ for all z such that $|z| < R$, then $a_n = b_n$.*

Example 4.3.1. *Find the domain of convergence of the series $\sum\left(\frac{iz-3}{5+2i}\right)^n$.*

Solution: *Here,* $u_n=\left(\frac{iz-3}{5+2i}\right)^n, u_{n+1}=\left(\frac{iz-3}{5+2i}\right)^{n+1}$

Now, $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim\limits_{n\to\infty}\left|\frac{iz-3}{5+2i}\right|=\frac{|iz-3|}{\sqrt{29}}$

$\Rightarrow$ *The given series is convergent, if* $\frac{|iz-3|}{\sqrt{29}}<1$

$\Rightarrow |iz-3| < \sqrt{29} \quad \Rightarrow |i||z-\frac{3}{i}| < \sqrt{29} \quad \Rightarrow |z+3i| < \sqrt{29}$

Thus, the given series is convergent in the circle $|z+3i| < \sqrt{29}$

Example 4.3.2. *Examine the behavior of the power series* $\sum \frac{z^{2n}}{(2n+3)^2}$ *on the circle of convergence.*

Solution: *Here* $u_n = \frac{1}{(2n+3)^2}$, $u_{n+1} = \frac{1}{(2n+5)^2} \Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2n+3}{2n+5}\right|^2 = 1$

Hence the radius of convergence of the given power series is 1 and center of convergence 0.

Now for every point on the circle of convergence, $\left|\frac{z^{2n}}{(2n+3)^2}\right| = \frac{|z|^{2n}}{|(2n+3)^2|} = \frac{1}{(2n+3)^2}$

[$\because$ $|z| = 1$ on circle of convergence]

Now, since by comparison test, $\sum \frac{1}{(2n+3)^2}$ *is convergent for all values of n, so the power series* $\sum \frac{z^{2n}}{(2n+3)^2}$ *is convergent for every point z on the circle of convergence.*

PRACTICE SET – I

Exercise 1. *The radius of convergence of power series* $\sum_{n=0}^{\infty}(4n^4 - n^3 + 3)z^n$ *is*

(a) 0 (b) 1

(c) 5 (d) ∞

Exercise 2. *The radius of convergence of the series* $\sum_{n=1}^{\infty}\frac{(z+2)^{n-1}}{(n+1)^3 4^n}$ *is*

(a) 2 (b) 1

(c) ∞ (d) 4

Exercise 3. *The power series* $\sum z^{n^2}$

(a) converges absolutely in unit circle centered at origin

(b) converges uniformly in unit circle centered at origin

(c) does not converge in unit circle

(d) converges in $|z| > 1$

Exercise 4. *The sequence* $\left\{ne^{-n^2}\right\}$ *is*

(a) convergent (b) divergent

(c) oscillating (d) convergent and converges to 0

Exercise 5. The series $\sum_{n=0}^{\infty}\frac{(3i)^{2n+1}}{(2n+1)!}$ is

(a) convergent (b) divergent
(c) constant (d) none of these

Exercise 6. The radius of convergence of the power series of the function $f(z)=\frac{1}{1-z}$ about $z=\frac{1}{4}$ is

(a) 1 (b) $\frac{1}{4}$
(c) $\frac{3}{4}$ (d) 0

Exercise 7. The series $\sum_{n=1}^{\infty}\frac{z^n}{n\sqrt{n+1}}$, $|z|\le 1$ is (GATE-2001)

(a) uniformly but not absolutely convergent
(b) uniformly and absolutely convergent
(c) absolutely convergent but not uniformly convergent
(d) convergent but not uniformly convergent

4.4. TAYLOR'S SERIES

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a + h$ be two points inside C.

Then $f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^n}{n!}f^n(a)+...$ ...(1)

or writing $z = a + h$, $h=z-a$, $f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+...+\frac{f^{(n)}(a)}{n!}(z\ a)^n+...$ ...(2)

Thus $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$, where $a_n=\frac{f^n(a)}{n!}$

This is called Taylor's theorem and the series (1) or (2) is called Taylor's series expansion of $f(a + h)$ or $f(z)$. The region of convergence of the series (2) is given by $|z - a|< R$, where the radius of convergence R is the distance from a to the nearest singularity of the function $f(z)$. On $|z-a|=R$, the series may or may not converge. For $|z-a|>R$, the series diverges.

If the nearest singularity of $f(z)$ is at infinity, then radius of convergence is infinite, i.e. the series converges for all z. If $a = 0$ in (1) or (2), the resulting series is often called Maclaurin series.

4.4.1. Some Special Series:

The following list show some special series together with their regions of convergence. In the case of multi-valued functions, the principal branch is used.

1. $e^z = 1 + z + \frac{z^2}{2!}+\frac{z^3}{3!}+...+\frac{z^n}{n!}+...$, $|z|<\infty$

2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \ldots, |z| < \infty$

3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} + \ldots, |z| < \infty$

4. $\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots, |z| < \infty$

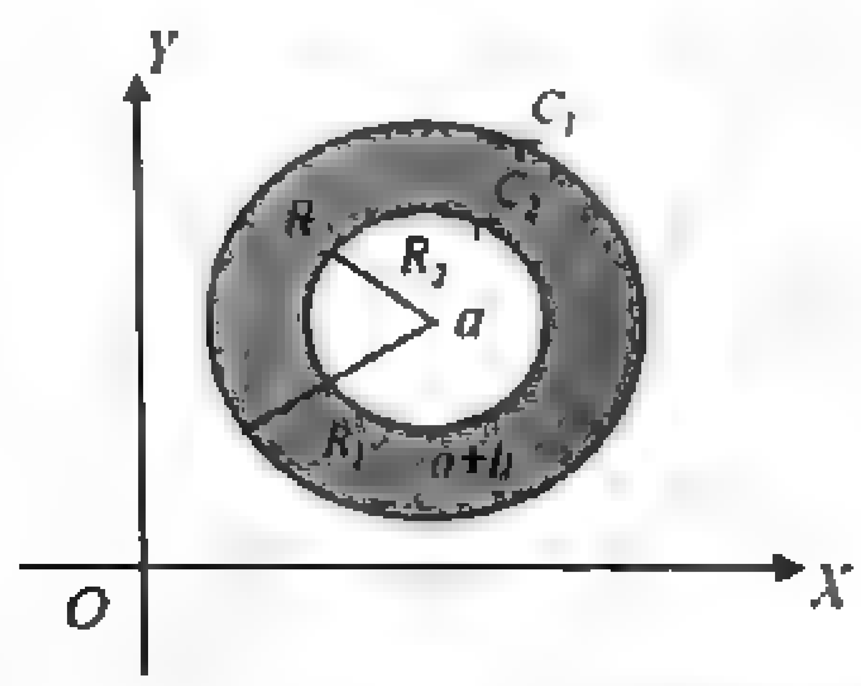
5. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots (-1)^{n-1} \frac{z^n}{n} + \ldots, |z| < 1$

6. $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots (-1)^{n-1} \frac{z^{2n-1}}{2n-1} + \ldots, |z| < 1$

7. $(1+z)^k = 1 + kz + \frac{k(k-1)}{2!} z^2 + \ldots + \frac{k(k-1)\ldots(k-n+1)}{n!} z^n + \ldots, |z| < 1$

Note: *A function f(z) can be expanded as a Taylor's series only if f(z) is analytic at all points inside the circle C. This assures the convergence of Taylor's series for f(z). If f(z) is not analytic at $z=z_0$, then f(z) is expanded about $z=z_0$ by Laurent's method.*

4.5. *LAURENT'S SERIES*



Let C_1 and C_2 be concentric circles of radii R_1 and R_2 respectively and centered at a. Suppose that f(z) is single-valued and analytic on C_1 and C_2 and in the annular region R between C_1 and C_2, shown shaded in the figure. Let a + h be any point in R. Then we have

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \ldots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \ldots, \quad \ldots(3)$$

where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$, $n = 0, 1, 2, \ldots$ and $a_{-n} = \frac{1}{2\pi i} \oint_{C_2} (z-a)^{n-1} f(z) dz$, $n = 1, 2, 3, \ldots$ ...(4)

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

In the above integrations we can replace C_1 and C_2 by any concentric circle C between C_1 and C_2. Then the coefficients (4) can be written in a single formula,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz, \quad n = 0, \pm 1, \pm 2, \ldots \quad \ldots(5)$$

By taking z=a+h, (3) can also be written as

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\ldots+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\ldots, \qquad \ldots(6)$$

where $a_n=\frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-a)^{n+1}}ds$, $n=0,\pm 1,\pm 2,\ldots$ ... (7)

(3) or (6) with coefficients (4), (5) or (7) is called Laurent series expansion.

The part $a_0+a_1(z-a)+a_2(z-a)^2+\ldots$ is called the **analytic part** of the Laurent series, while the remainder of the series which consists of inverse powers of $z-a$ is called the **principal part**.

Note: If the principal part is zero, then the Laurent series reduces to the Taylor series.

PRACTICE SET – II

Exercise 1. Let $f(z)=u+iv$ be an entire function having Taylor's series expansion as $\sum_{n=0}^{\infty}a_nz^n$. If $f(x)=u(x,0)$ and $f(iy)=iv(0,y)$, then

(a) $a_{2n}=0\ \forall n$ (b) $a_0=a_1=a_2=a_3=0,\ a_4\neq 0$

(c) $a_{2n+1}=0\ \forall n$ (d) $a_0\neq 0$ but $a_2=0$

Exercise 2. Let f be an analytic function and let $f(z)=\sum_{n=0}^{\infty}a_n(z-2)^{2n}$ be Taylor's series of $f(z)$ in some disc. Then

(a) $f^{(n)}(0)=(2n)!a_n$ (b) $f^{(n)}(2)=n!a_n$

(c) $f^{(2n)}(2)=(2n)!a_n$ (d) $f^{(2n)}(2)=n!a_n$

Exercise 3. The value of a_4 in the expansion of e^{-z} about $z=\frac{\pi}{4}$ is

(a) $e^{\frac{\pi}{4}}$ (b) $e^{-\frac{\pi}{4}}$

(c) $\frac{e^{\frac{\pi}{4}}}{4!}$ (d) $\frac{e^{-\frac{\pi}{4}}}{4!}$

Exercise 4. Expand $\sin^3 z$ about $z=0$.

Exercise 5. In Laurent series expansion of $f(z)=\frac{1}{z-1}-\frac{1}{z-2}$ valid in region $|z|>2$, the coefficient of $\frac{1}{z^2}$ is **(GATE-2004)**

(a) -1 (b) 0

(c) 1 (d) 2

KEY POINTS

- *A series having the form* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *is called a power series centered at* z_o
- ***Cauchy-Hadamard theorem:*** *It states that* $\exists$ *a positive number R such that series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges absolutely for* $|z-z_0| < R$, *converges uniformly for* $|z-z_o| \le \rho < R$ *for every* $\rho < R$ *and diverges for* $|z-z_0| > R$, *where* $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, *R is called the radius of convergence.*
- *Let f(z) be analytic inside and on a simple closed curve C and 'a' be a point inside C, then* $f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$ *is the Taylor's series expansion of f in the neighbourhood of point a.*
- $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots$, *where* $a_n = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-a)^{n+1}}ds$, $n = 0, \pm 1, \pm 2, \ldots$ *is Laurent series expansion.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *The radius of convergence of* $\sum_{n=0}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2}}{n^3}z^n$ *is* ***(GATE-2006)***

(a) e *(b) 1/e* *(c) 1* *(d)* ∞

Solution: (b) *Radius of convergence is given by*

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^{n^2}}{n^3}\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(n^{1/n}\right)^3} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{\lim_{n\to\infty}\left((n)^{1/n}\right)^3} = e$$

$$\Rightarrow R = \frac{1}{e}$$

Example 2. *It is given that* $\sum_{n=0}^{\infty} a_n z^n$ *converges at* $z = 3+4i$. *Then the radius of convergence of the power series* $\sum_{n=0}^{\infty} a_n z^n$ *is* **(GATE-2007)**

(a) ≤ 5 (b) ≥ 5 (c) < 5 (d) > 5

Solution: (b) *If* $\sum_{n=1}^{\infty} a_n z^n$ *has radius of convergence r, then the power series converges for* $|z| < r$ *and diverges for* $|z| > r$. *At* $|z| = r$ *series may or may not converge. Given that it converges for* $z = 3 + 4i$ *i.e. for a point z with* $|z| = 5$, *i.e., the point* $z = 3 + 4i$ *either lie inside the circle or lies on the boundary.*

$\Rightarrow$ *Either r is greater than 5 or equal to 5 i.e.* $r \geq 5$.

Example 3. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an arbitrary analytic function satisfying* $f(0) = 0$ *and* $f(1) = 2$. *Then* **(GATE-2007)**

(a) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| > n$

(b) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| < n$

(c) there exists a bounded sequence $\{z_n\}$ such that $|f(z_n)| > n$

(d) there exists a sequence $\{z_n\}$ such that $z_n \to 0$ and $f(z_n) \to 2$

Solution: (a) *Given that* $f(z)$ *is analytic function on* $\mathbb{C}$ *satisfying* $f(0) = 0$ *and* $f(1) = 2$

$\Rightarrow f(z)$ *is non- constant, entire function* $\Rightarrow f(z)$ *is unbounded [By Liouville's Theorem]*

$\therefore \exists$ *sequence* $\{z_n\}$ *such that* $|z_n| > n$ *and* $|f(z_n)| > n$

Example 4. *The coefficient of* $(z-\pi)^2$ *in the Taylor series expansion of* $f(z) = \begin{cases} \dfrac{\sin z}{z-\pi} & \text{if } z \neq \pi \\ -1 & \text{if } z = \pi \end{cases}$ *around* π *is* **(GATE-2013)**

(a) 1/2 (b) -1/2 (c) 1/6 (d) -1/6

Solution: (c) *Taylor's series expansion of* $f(z) = \begin{cases} \dfrac{\sin z}{z-\pi} & \text{if } z \neq \pi \\ -1 & \text{if } z = \pi \end{cases}$ *is given by*

$$f(z) = \frac{\sin z}{z-\pi} = \frac{\sin(z-\pi+\pi)}{z-\pi}$$

$$= \frac{\sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi}{(z-\pi)} = -\left(\frac{\sin(z-\pi)}{z-\pi}\right)$$

$$= -\left(\frac{(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \ldots}{(z-\pi)}\right)$$

$\Rightarrow$ *Coefficient of* $(z-\pi)^2 = \frac{1}{6}$

Example 5. *The radius of convergence of the power series* $\sum_{n=0}^{\infty} 4^{(-1)^n n} z^{2n}$ *is* ________ **(GATE-2014)**

Solution: (Ans. 0.5) *Consider, the power series* $\sum_{n=0}^{\infty} 4^{(-1)^n n} z^{2n}$, *here* $a_n = 4^{(-1)^n . n}$

$$a_n = \begin{cases} 4^{-n} & , \quad n \text{ odd} \\ 4^{n} & , \quad n \text{ even} \end{cases}$$

If n is odd, $\frac{1}{R} = \lim_{n\to\infty} |4^{-n}|^{\frac{1}{n}} = 4^{-1} = \frac{1}{4}$, *where R is the radius of convergence*

$\therefore\ R = 4 \quad \Rightarrow z^2 < 4 \quad \Rightarrow |z| < 2$

If n is even, $\frac{1}{R} = \lim_{n\to\infty} |4^{n}|^{\frac{1}{n}} \quad \Rightarrow R = \frac{1}{4}$

$z^2 < \frac{1}{4} \quad \Rightarrow |z| < \frac{1}{2}$

$\Rightarrow$ *Radius of convergence of the Power series is* $\frac{1}{2} = 0.5$.

Example 6. *If the power series* $\sum_{n=0}^{\infty} a_n (z+3-i)^n$ *converges at* $5i$ *and diverges at* $-3i$, *then the power series* **(GATE-2015)**

(a) *converges at* $-2+5i$ *and diverges at* $2-3i$

(b) *converges at* $2-3i$ *and diverges at* $-2+5i$

(c) *converges at both* $2-3i$ *and* $-2+5i$

(d) *diverges at both* $2-3i$ *and* $-2+5i$

Solution: (a) *Consider the power series* $\sum_{n=0}^{\infty} a_n (z-(-3+i))^n$

As, $|5i-(-3+i)| = |3+4i| = 5$ *and* $|-3i-(-3+i)| = |3-4i| = 5$

Also given that the series converges at 5i and diverges at -3i. Therefore, the radius of convergence of $\sum_{h=0}^{\infty} a_n (z-(-3+i))^n$ *is 5 and 5i and -3i lies on the boundary of the circle of convergence.*

At $-2+5i$, $|-2+5i-(-3+i)| = |4i+1| = \sqrt{17} \sim 4$ *(approximately)*

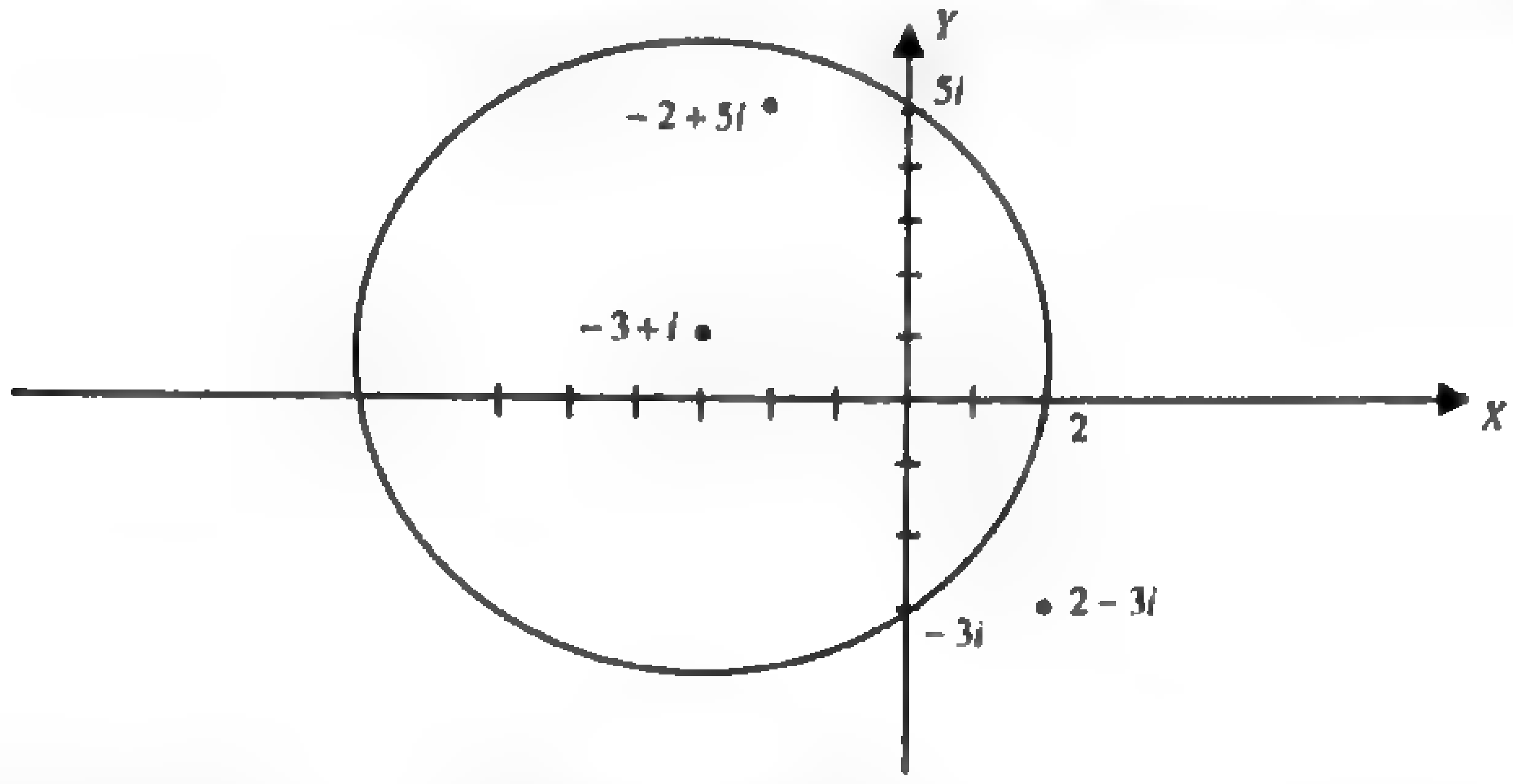
Hence, the given power series converges at $-2+5i$

At $2-3i$, $|2-3i-(-3+i)| = |5-4i| = \sqrt{41} \sim 6$ *(approximately)*

Hence, the given power series diverges at $2-3i$

OR

As -2 + 5i lies inside the circle of convergence and 2-3i lies outside the circle of convergence.



∴ *The series converges at -2 + 5i and diverges at 2 – 3i.*

Example 7. *The maximum value of the function f(x,y,z)=xyz subject to the constraint xy+yz+zx-a=0, a>0 is* ***(GATE-2012)***

(a) $a^{\frac{3}{2}}$ (b) $(a/3)^{3/2}$ (c) $(3/a)^{3/2}$ (d) $(3a/2)^{3/2}$

Solution: (b)

$x, y, z > 0$

By arithmetic-geometric inequality $\frac{a}{3}=\frac{xy+xz+yz}{3}\geq\sqrt[3]{xy.xz.yz}=(f(x,y,z))^{\frac{2}{3}}$

$$\Rightarrow f(x,y,z)\leq\left(\frac{a}{3}\right)^{\frac{3}{2}}$$

At x=y =z, $f(x,y,z)=x^3$

As x=y=z also satisfies the constraint. Therefore, $a=3x^2$.

Note that $f(x,y,z)=x^3=\left(\frac{a}{3}\right)^{\frac{3}{2}}$

So maximum value is $\left(\frac{a}{3}\right)^{\frac{3}{2}}$

Example 8. *The power series* $\sum_{0}^{\infty}2^{-n}z^{2n}$ *converges if* ***(CSIR UGC NET JUNE-2011)***

(a) $|z|\leq 2$ (b) $|z|<2$ (c) $|z|\leq\sqrt{2}$ (d) $|z|<\sqrt{2}$

Solution: (d) *Given Power Series is* $\sum_{0}^{\infty}2^{-n}z^{2n}$

Let $z^2=t$ $\therefore \sum_0^{\infty} 2^{-n} z^{2n} = \sum_0^{\infty} 2^{-n} t^n$

Here, $a_n = 2^{-n}$, $a_{n+1} = 2^{-n-1}$

By Cauchy-Hadamard theorem

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left|\frac{2^{-n-1}}{2^{-n}}\right| \Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left|\frac{1}{2}\right| \quad \Rightarrow R = 2$$

Power series converges inside the circle of convergence, i.e., $|t| < R$

$\Rightarrow |z^2| < R$, *i.e., Power series converges if* $|z^2| < 2$ $\Rightarrow$ *Power series converges if* $|z| < \sqrt{2}$

Now check for $|z| = \sqrt{2}$

For $z = \sqrt{2}$, $\sum_0^{\infty} 2^{-n} 2^n = \sum_0^{\infty} 1$ *diverges* $\therefore$ *Power series converges if* $|z| < \sqrt{2}$

$\Rightarrow$ *Option (d) is correct.*

Example 9. *Consider the power series* $\sum_{n\ge1} a_n z^n$, *where* a_n = *number of divisors of* n^{50}. *Then the radius of convergence of* $\sum_{n\ge1} a_n z^n$ *is* ***(CSIR UGC NET DEC-2011)***

(a) 1 *(b) 50* *(c) 1/50* *(d) 0*

Solution: (a) *We know that d(n) ~ n, for large n [d(n) = Number of divisors of n].*

$\therefore$ *Radius of convergence of the power series* $\sum_{n\ge1} d(n^{50}) z^n$ *will be same as that of the series* $\sum_{n\ge1} n^{50} z^n$

Here, $a_n = n^{50}$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{50}}{n^{50}}\right| = 1 \Rightarrow R = 1$$

$\therefore$ *Radius of convergence of the power series* $\sum_{n\ge1} d(n^{50}) z^n$ *is also 1.*

Example 10. *Consider the power series* $\sum_{n=1}^{\infty} z^{n!}$. *The radius of convergence of this series is*

(CSIR UGC NET DEC-2012)

(a) 0 *(b)* ∞ *(c) 1* *(d) a real number greater than 1*

Solution: (c) $\sum z^{n!} = z + z^2 + z^6 + z^{24} + z^{120} + \ldots$

We know, $\sum z^n = 1 + z + z^2 + \ldots$, *converges for* $|z| < 1$, *i.e., its radius of convergence is 1*

$\sum z^{n!}$ *contains less terms than* $\sum z^n$

$\therefore$ *Radius of convergence of* $\sum z^{n!}$ *is* ≥ 1.

But at z = 1, the series $\sum z^{n!}$ *diverges* $\therefore$ *Radius of convergence of* $\sum z^{n!}$ *is also 1.*

$\Rightarrow$ *option (c) is correct.*

Example 11. *Let p(x) be a polynomial of the real variable x of degree $k \geq 1$. Consider the power series f (z) = $\sum_{n=0}^{\infty} p(n)z^n$, where z is a complex variable. Then the radius of convergence of f (z) is*

(CSIR UGC NET JUNE-2014)

(a) 0 (b) 1 (c) k (d) ∞

Solution: (b) *Let p(x) = x^2 be a polynomial of the real variable x of degree x = 2*

Then Power Series $f(z) = \sum_{n=0}^{\infty} p(n)z^n = \sum_{n=0}^{\infty} n^2 z^n$

Take $a_n = n^2$

Radius of convergence of $f(z)$ *is* $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\Rightarrow \frac{1}{R} = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} \quad \Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \quad \Rightarrow \frac{1}{R} = 1 \quad \Rightarrow R = 1$$

So options (a), (c) and (d) are incorrect

$\Rightarrow$ *option (b) is correct.*

Example 12. *Let* $\sum_{n=1}^{\infty} a_n z^n$ *be a convergent power series such that* $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = R > 0$. *Let p be a polynomial of degree d. Then the radius of convergence of the power series* $\sum_{n=0}^{\infty} p(n)z^n$ *equals*

(CSIR UGC NET DEC-2014)

(a) R (b) d (c) Rd (d) R+d

Solution: (a) *Take d = 2 and p(n) = n^2*

p(n+1) = $(n+1)^2$

Radius of convergence of the Power series $\sum_{n=0}^{\infty} n^2 z^n$ *is given by* $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1$

$\Rightarrow$ *only option (a) is correct.*

Example 13. *Consider the following power series in the complex variable z:*

$f(z) = \sum_{n=1}^{\infty} n\log n\, z^n, g(z) = \sum_{n=1}^{\infty}\frac{e^{n^2}}{n}z^n$. *If r, R are the radii of convergence of f and g respectively, then*

(a) r = 0, R = 1. (b) r = 1, R = 0. (c) r = 1, R = ∞. (d) r = ∞, R = 1.

Solution: (b) $f(z) = \sum_{n=1}^{\infty} n\log n\, z^n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\log(n+1)}{n\log n} = \left(1+\frac{1}{n}\right)\frac{\log n\left(1+\frac{1}{n}\right)}{\log n} = \left(1+\frac{1}{n}\right)\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}$$

$$=\left(1+\frac{1}{n}\right)\left[1+\frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right]=\left(1+\frac{1}{n}\right)\left[1+\frac{1}{n\log n}-\frac{1}{2n^2\log n}+\ldots\right]$$

$$\Rightarrow\frac{1}{r}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1.1=1\Rightarrow r=1$$

Now $g(z)=\sum_{n=1}^{\infty}\frac{e^{n^2}}{n}z^n$

$$\frac{1}{R}=\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{e^{n^2}}{n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\frac{|e^n|}{|n|^{\frac{1}{n}}}=\infty\left[\because\lim_{n\to\infty}|n|^{\frac{1}{n}}=1\right]$$

$\Rightarrow R=0\Rightarrow$ *option (b) is correct.*

Example 14. *The radius of convergence of the series* $\sum_{n=1}^{\infty}z^{n^2}$ *is* ***(CSIR UGC NET DEC-2016)***

(a) 0 *(b)* ∞ *(c) 1* *(d) 2*

Solution: *(c) Given series is* $\sum_{n=1}^{\infty}z^{n^2}$

Clearly at z=1, the series $\sum_{n=1}^{\infty}1$ *diverges*

$\Rightarrow$ *Options (b) and (d) are incorrect.*

Now, consider a point $z=\frac{1}{6}$

$$\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n^2}=\frac{1}{6}+\frac{1}{6^4}+\frac{1}{6^9}+\ldots$$

As we know $\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n}$ *is convergent* $\Rightarrow\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n^2}$ *is convergent*

$\Rightarrow$ *Option (a) is incorrect and hence (c) is correct*

ASSIGNMENT - 4.1

NOTE: CHOOSE THE BEST OPTION

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ have the radius of convergence L and L' respectively, then
   (a) $L = L'$ (b) $L > L'$ (c) $L < L'$ (d) none of these

2. Let $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^n$ and $g(z) = \sum_{n=0}^{\infty} a_n z^n$ have the radius of convergence L and L' respectively, then
   (a) $L > L'$ (b) $L < L'$ (c) $L = L'$ (d) none of these

3. A Maclaurin series is a Taylor series with centre
   (a) $z_0 = 1$ (b) $z_0 = 0$ (c) $z_0 = 2$ (d) none of these

4. If f(z) is an entire function, then the Taylor series is
   (a) convergent for all z (b) divergent for all z (c) constant (d) none of these

5. The series $\sum a_n$ converges absolutely if
   (a) $\sum |a_n|$ converges (b) $\sum a_n$ converges (c) $\sum |a_n|$ diverges (d) none of these

6. The centre of convergence for the power series $\sum_{n=0}^{\infty} (n+2i)^n z^n$ is
   (a) 0 (b) $-i$ (c) $-2i$ (d) none of these

7. The centre of convergence for the power series $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$ is
   (a) $-i\sqrt{2}$ (b) $-i$ (c) $\sqrt{2}$ (d) none of these

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

8. If series $\sum a_n$ converges absolutely, then
   (a) $\sum a_n$ converges (b) $\sum a_n$ does not converges
   (c) $\sum a_n$ diverges (d) $\sum |a_n|$ converges

9. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is not
   (a) divergent (b) convergent (c) constant (d) equal to zero

10. The centre of convergence for the power series $\sum_{n=0}^{\infty} \frac{n+5i}{(2n)!}(z-i)^n$ is not
   (a) i (b) $-i$ (c) 2 (d) all of these

ASSIGNMENT - 4.2

NOTE: CHOOSE THE BEST OPTION

1. $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ *is equal to*
   (a) e
   (b) e^{-z}
   (c) e^z
   (d) none of these

2. $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$ *is equal to*
   (a) sin z
   (b) cos z
   (c) tan z
   (d) none of these

3. $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$ *is equal to*
   (a) sin z
   (b) cos z
   (c) tan z
   (d) none of these

4. *The radius of convergence of the power series* $\sum_{n=0}^{\infty}(n+2i)^n\ z^n$ *is*
   (a) 0
   (b) one
   (c) ∞
   (d) none of these

5. *The radius of convergence of the power series* $\sum_{n=0}^{\infty}\frac{n+5i}{(2n)!}\ (z-i)^n$ *is*
   (a) 0
   (b) ∞
   (c) one
   (d) none of these

6. *The radius of convergence of the power series* $\sum_{n=0}^{\infty}\frac{(z-2i)^n}{n^n}$ *is*
   (a) ∞
   (b) zero
   (c) one
   (d) none of these

7. *The centre of convergence for the power series* $\sum_{n=0}^{\infty}\left(\frac{4-2i}{1+5i}\right)^n z^n$ *is*
   (a) 1
   (b) 2
   (c) zero
   (d) none of these

8. $\left\{\frac{e^{in\pi/4}}{n}\right\}_{n=1}^{\infty}$ is

(a) unbounded
(b) bounded
(c) divergent
(d) none of these

9. The sequence $\{(-1)^n + 100i\}_{n=1}^{\infty}$ is

(a) unbounded
(b) bounded
(c) convergent
(d) none of these

10. The series $\sum_{n=1}^{\infty}\frac{(3i)^n n!}{n^2}$ is

(a) convergent
(b) divergent
(c) constant
(d) none of these

11. For the power series $\sum \frac{n!}{n^n} z^n$, the radius of convergence is

(a) e
(b) 1
(c) ∞
(d) zero

12. If $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$, then

(a) For $n \leq 0$, $a_n = \frac{1}{n!}f^{(n)}(a)$
(b) For $n \geq 0$, $a_n = 0$
(c) For $n \geq 0$, $a_n = \frac{1}{n!}f^{(n)}(a)$
(d) None of these

13. The radius of convergence of power series $f(z) = \sum \frac{1}{n^p} z^n$ is

(a) 1
(b) 2
(c) 3
(d) 0

14. The power series $\sum_{n=1}^{\infty}\frac{z^n}{n^a}, a \in \mathbb{R}$

(a) has radius of convergence 1
(b) cannot say about radius of convergence unless the value of a is specified
(c) diverges for all values of a
(d) diverges only for specified value of a

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

15. The radius of convergence of the power series $\sum \frac{2^{-n} z^n}{1+in^2}$ is not

(a) 2 (b) 1/2
(c) 1 (d) 4

16. The series $\sum_{n=1}^{\infty} n^2 \left(\frac{i}{2}\right)^n$ is

(a) convergent (b) divergent
(c) bounded (d) unbounded

17. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges at

(a) for all z (b) $z = 1$
(c) $z = 2$ (d) None of these

18. The series $\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!}$ is

(a) convergent (b) divergent
(c) non-zero (d) absolutely convergent

19. The sequence $\left\{\frac{n\pi i}{n+i}\right\}_{n=1}^{\infty}$ is

(a) unbounded (b) divergent
(c) convergent (d) bounded

20. The sequence $\left\{\frac{n\pi}{1+3ni}\right\}_{n=1}^{\infty}$ is

(a) convergent (b) divergent
(c) unbounded (d) bounded

ASSIGNMENT - 4.3

NOTE: CHOOSE THE BEST OPTION

1. *The radii of convergence of the power series* $\sum a_n z^n$ *and* $\sum b_n z^n$ *are* R_1 *and* R_2. *The radius of convergence R of power series* $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ *satisfies*

   (a) $R = R_1 + R_2$ (b) $R \geq \min\{R_1, R_2\}$

   (c) $R \leq \min\{R_1, R_2\}$ (d) none of these

2. *Let f and g be analytic in a domain D, and let each have zeros of order m and n at* z_0, *respectively, then order of zeros of f + g is*

   (a) $m + n$ (b) $\geq \min\{m, n\}$

   (c) $\leq \max\{m, n\}$ (d) mn

3. *The sum of the power series* $\sum_{n=1}^{\infty} z^n / n$ *for* $|z| < 1$, *is*

   (a) $\log(1 + z)$ (b) $-\log(1 - z)$

   (c) $\log(1 - z)$ (d) none of these

4. *The sum of the power series* $\sum_{n=0}^{\infty} z^{2n+1}/(2n+1)$ *for* $|z| < 1$, *is*

   (a) $(1/2)\log[(1 + z)/(1-z)]$ (b) $-\log(1 - z)$

   (c) $-(1/2)\log[(1 + z)/(1-z)]$ (d) none of these

5. *The sum of the power of series* $\sum_{n=0}^{\infty}(n+3)z^n$ *for* $|z| < 1$ *is*

   (a) $(3-2z)(1-z)^{-2}$ (b) $(3+2z)(1-z)^{-2}$

   (c) $(3-2z)(1+z)^{-2}$ (d) none of these

6. *Consider the series* $\sum_{n=1}^{\infty} a_n$, *where* $a_{2n-1} = \left(\frac{1+i}{3\sqrt{2}}\right)^{2n-1}$ *and* $a_{2n} = \left(\frac{1-i}{2\sqrt{2}}\right)^{2n}$, *then* $\sum a_n$

   (a) diverges (b) converges

   (c) may or may not converge (d) constant

7. *Function* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in the domain* $|z| > 3$ *has a Laurent series expansion given by*

   (a) $\frac{1}{z^2} - \frac{2}{z^3} + \frac{13}{z^4} - \frac{20}{z^5} + \ldots$ (b) $\frac{1}{z^2} - \frac{8}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \ldots$

(c) $\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\ldots$
(d) $\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{20}{z^5}+\ldots$

8. *The coefficient of* $\frac{1}{z}$ *in the Laurent series expansion of* $\frac{\sin 2z}{z^3}$ *is*

(a) 0
(b) 2
(c) -1
(d) 1

9. *The radius of convergence of the series given as* $1+z+\frac{z^2}{2}-\frac{z^3}{3!}+\frac{z^4}{2^2}+\frac{z^5}{5!}+\frac{z^6}{2^3}-\frac{z^7}{7!}+\ldots$, *is*

(a) 2
(b) $\sqrt{2}$
(c) ∞
(d) 0

10. *The domain of convergence of the series* $\sum_{n=0}^{\infty} n^2\left(\frac{z^2+1}{1+i}\right)^n$ *is*

(a) $|z+1|<\sqrt{2}$
(b) $|z+1|<2$
(c) $|z^2+1|<\sqrt{2}$
(d) $|z^2+1|<2$

11. *Taylor's expansion of* $f(z)=\frac{1}{(z+1)^2}$ *about the point* $z=-i$ *is*

(a) $(2i-2)+\frac{1}{z}+\frac{1}{z+1}$
(b) $\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$
(c) $\frac{i}{2}\left[1-\sum_{n=1}^{\infty}(-1)^n\frac{(n-1)(z+i)^n}{(1-i)^n}\right]$
(d) *none of the above*

12. *When* $0<|z|<4$, *the expansion of* $\frac{1}{4z-z^2}$ *is*

(a) $\sum_{n=0}^{\infty}\frac{z^{n+1}}{4^{n+1}}$
(b) $\sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{4^{n+1}}$
(c) $\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$
(d) *none of these*

13. *If* $\sin z=\sum_{n=0}^{\infty} a_n (z-\pi/4)^n$, *then* a_6 *equals*

(a) 0
(b) $\frac{1}{720}$
(c) $\frac{1}{(720\sqrt{2})}$
(d) $\frac{-1}{(720\sqrt{2})}$

14. *Coefficient of 1/z in expansion of* $\log\left(\frac{z}{z-1}\right)$, *valid in* $|z|>1$, *is*

(a) -1
(b) 1
(c) $-\frac{1}{2}$
(d) $\frac{1}{2}$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

15. *The radius of convergence of the power series* $f(z)=\sum\frac{n+1}{(n+2)(n+3)}z^n$ *is not*

(a) 1
(b) 2
(c) 3
(d) 4

16. *If* $f(z)=\frac{1}{(z-1)(z-2)}$, *then expansion(s) of f(z) is/are*

(a) $\sum_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)z^{n-1};|z|<1$
(b) $\sum_{n=0}^{\infty}\frac{z^{n-1}}{2^n};|z|<1$
(c) $...+7z^{-4}+3z^{-3}+z^{-2};|z|>2$
(d) only (a) is true

ANSWERS TO EXERCISES

(PRACTICE SET – I)

Exercise 1: *(b)* **Exercise 2:** *(d)* **Exercise 3:** *(a)* **Exercise 4:** *(a,d)*
Exercise 5: *(a)* **Exercise 6:** *(c)* **Exercise 7:** *(b)*

(PRACTICE SET – II)

Exercise 1: (a) *Exercise 2: (c)* *Exercise 3: (d)* *Exercise 4:* $z^3+\frac{1}{2}z^5+\frac{13}{120}z^7+\ldots$

Exercise 5: (a)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 4.1

1.(a)	2. (c)	3. (b)	4. (a)	5. (a)	6. (a)	7. (a)
8. (a)	9. (b,c,d)	10. (b,c,d)				

ASSIGNMENT - 4.2

1. (c)	2. (b)	3. (a)	4. (a)	5. (b)	6. (a)	7. (c)
8. (b)	9. (b)	10. (b)	11. (a)	12. (c)	13. (a)	14. (a)
15. (b,c,d)	16. (a,c)	17. (a,b,c)	18. (a,c,d)	19. (c,d)	20. (a,d)	

ASSIGNMENT - 4.3

1. (b)	2. (b)	3. (b)	4. (a)	5. (a)	6. (b)	7. (c)
8. (b)	9. (b)	10. (c)	11. (b)	12. (c)	13. (d)	14. (b)
15. (b,c,d)	16. (a,c)					

CHAPTER - 5
CLASSIFICATION OF SINGULARITIES AND LAURENT SERIES

INTRODUCTION

We have studied that singularity is in general a point at which given mathematical object is not defined or a point of an exceptional set where it fails to well-behaved in some particular way. In this chapter, we shall discuss types of singularities and calculus of residues at those singularities. Also we will study Laurent series. The Laurent series of a complex function f(z) is a representation of that function as a power series which includes terms of negative degree. It may be used in cases where a Taylor series expansion cannot be applied. So, we will extend Taylor series to Laurent series which is found about a singular point in this chapter.

5.1. ZEROS AND SINGULAR POINTS

5.1.1. Zeros of an Analytic Function:

The value of z for which the analytic function f(z) becomes zero is said to be the zero of f(z).

If f(z) is analytic in a domain D and z_0 is any point of D, then we can expand f(z) as Taylor series about $z = z_0$ given by $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$

If $a_0=a_1=a_2=...=a_{m-1}=0$ *and* $a_m \neq 0$, *f(z) is said to have a zero of order m at $z=z_0$.*

A zero of order one (m=1) is said to be a simple zero.

Definition: *An analytic function defined in $|z-a|<R$ is said to have zero of order $m \in \mathbb{N}$ at point a if f(a) =* $f'(a)=...=f^{(m-1)}(a)=0$ *and* $f^{(m)}(a)\neq 0$.

Note: *Zeros of an analytic function are always isolated.*

For Example: *The function* $f(z)=\frac{z-2}{z^2}\sin\left(\frac{1}{z-1}\right)$ *has zero at z=2, because at z=2, f(z)=0*

Also, zeros are obtained when $\sin\left(\frac{1}{z-1}\right)=0$

$$\Rightarrow \frac{1}{z-1}=n\pi \quad \Rightarrow z-1=\frac{1}{n\pi} \quad \Rightarrow z=1+\frac{1}{n\pi},\ n=\pm 1, \pm 2, ...$$

5.1.2. Branch Points and Branch Lines:

Consider the function $w=f(z)=z^{\frac{1}{2}}. \quad \Rightarrow w=\sqrt{r}e^{\frac{i\theta}{2}}\ \left[\because z=re^{i\theta}\right]$

Let z starts from $z_0=re^{i\theta_1}$ *and moves along a circle of radius r centred at origin. Thus, at z_0*

$$w=\sqrt{r}e^{\frac{i\theta_1}{2}} \qquad ...(1)$$

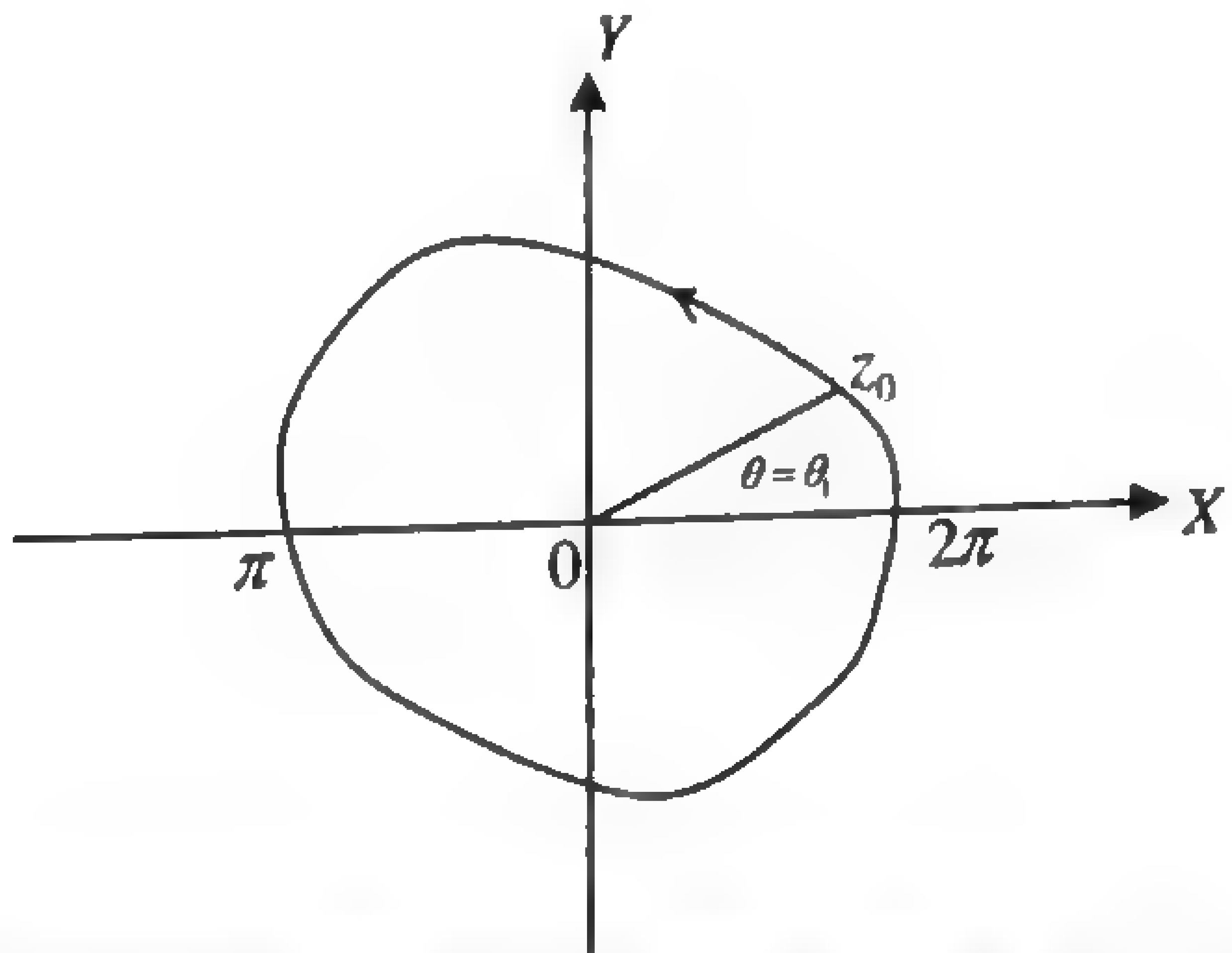
Now after traversing $|z| = r$ in anti-clockwise direction, when z reaches point z_0, then at z_0

$w=\sqrt{r}e^{\frac{i(\theta_1+2\pi)}{2}}=-\sqrt{r}e^{\frac{i\theta_1}{2}}$ ...(2)

Thus the same value of w is not achieved at $z = z_0$.

However, when z traverse $|z| = r$ second time then at z_0, $w=\sqrt{r}e^{\frac{i(\theta_1+4\pi)}{2}}=\sqrt{r}e^{\frac{i\theta_1}{2}}$

Thus, the same value of w has been obtained as in (1). Hence 0 is the Branch point of the function $w=z^{\frac{1}{2}}$



Basically, branch points are the points where the various sheets of a multiple valued function come together. The branches of the function are the various sheets of the function. For e.g., if $0 \le \theta < 2\pi$, then one branch of the multiple-valued function $z^{\frac{1}{2}}$ is obtained and if $2\pi \le \theta < 4\pi$ then the other branch of the function is obtained. But it is clear that each branch of the function is single-valued. In order to keep the function single-valued, a branch cut or branch line is set up. A branch cut is a curve in the complex plane such that it is possible to define a single analytic branch of a multi-valued function on the plane minus that curve.

It should also be noted that any circuit around any point except $z = 0$ does not yield different values of function $z^{\frac{1}{2}}$. Thus $z = 0$ is the only finite branch point.

Definition: *The point z_0 is called a branch point for the complex (multiple) valued function f(z) if the value of f(z) does not return to its initial value as a closed curve starting from some arbitrary point on the curve, around the point is traced in such a way that f varies continuously as the path is traced.*

Note: *Branch Points of multiple-valued functions are singular points.*

Example 1: *$f(z) = (z-4)^{1/3}$ has a branch point at $z = 4$.*

Example 2: *$f(z) = \ln(z^2 - z - 2)$ has branch points where $z^2 - z - 2 = 0$, i.e., at $z = -1$ and $z = 2$.*

5.1.3. *Singular Point or Singularity:*

Let f(z) be a complex-valued function. A point $z = z_0$ is called a singular point or singularity of f(z), if f(z) is not analytic at $z = z_0$, but every neighbourhood of z_0 contains at least one point at which f is analytic. Singularities are of two type-isolated and non-isolated singularities.

5.1.3.1. Isolated and Non – Isolated Singularities: *The point $z = z_0$ is called an isolated singularity or isolated singular point of $f(z)$ if there exists $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e., there exists a deleted δ neighbourhood of z_0 containing no singularity). If no such δ can be found, then z_0 is called a non-isolated singularity. If z_0 is not a singular point and there exists find $\delta > 0$ such that $|z - z_0| = \delta$ encloses no singular point, then z_0 is called an ordinary point of $f(z)$.*

e.g. (1) $f(z) = \dfrac{1}{z-1}$, $z=1$ is an isolated singularity of $f(z)$ in $|z-1| < 5$

(2) $f(z) = \log z$, $z=0$ is non-isolated singularity of $f(z)$.

Remark: *If a function has finite number of singular points, then singularity at those points is always isolated.*

5.1.4 Types of Singularities:

(1) Removable Singularity: *If all the coefficients b_n in the Laurent series expansion of $f(z)$ about z_0 are zero, i.e., if the principal part of $f(z)$ at $z=z_0$ consists of no terms, then z_0 is called removable singularity of $f(z)$. If f has a removable singularity at $z=z_0$, then there exists an analytic function $g(z)$ at z_0 such that $f(z)=g(z)$ for all z in some deleted neighbourhood of z_0.*

For Example: *If $f(z)=\dfrac{e^z-1}{z}$, then $z=0$ is a removable singularity since $f(0)$ is not defined but $\lim\limits_{z\to0}\dfrac{e^z-1}{z}=1$.*

We define $f(0)=\lim\limits_{z\to0}\dfrac{e^z-1}{z}=1$.

$$\frac{e^z-1}{z}=\frac{1}{z}\left[\left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\dots\right)-1\right]=1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots$$

Since, the laurent series expansion of $f(z)=\dfrac{e^z-1}{z}$ about $z=0$ has no principal part, so $z=0$ is a removable singularity of $f(z)$.

Remark: *If f has an isolated singularity at z_0, then $z=z_0$ is a removable singularity iff one of the following conditions hold:*

(i) *f is bounded in a deleted neighbourhood of z_0,*

(ii) *$\lim\limits_{z\to z_0} f(z)$ exists,*

(iii) *$\lim\limits_{z\to z_0}(z-z_0)f(z)=0$*

(2) Pole: *If in the Laurent series expansion of $f(z)=\sum\limits_{n=0}^{\infty}a_n(z-a)^n$ in some deleted neighbourhood of a, the principal part has only a finite number of terms given by $\dfrac{a_{-1}}{z-a}+\dfrac{a_{-2}}{(z-a)^2}+\dots+\dfrac{a_{-n}}{(z-a)^n}$, where $a_{-n}\neq 0$,*

then $z = a$ *is called a pole of order n. If* $n = 1$, *it is called a simple pole. If f(z) has a pole at* $z = a$, *then* $\lim_{z\to a} f(z) = \infty$

Example 1. $f(z)=\dfrac{1}{(z-3)^4}$ *has a pole of order 4 at z=3.*

Example 2. $f(z)=\dfrac{3z-2}{(z-1)^2(z+1)(z-4)}$ *has a pole of order 2 at z=1 and simple pole at z=-1 and z= 4.*

Remark:

(1) *If* $g(z) = (z - z_0)^n f(z)$, *where* $f(z_0)\neq 0$ *and n is a positive integer, then* $z = z_0$ *is called a zero of order n of g(z). If* $n = 1$, z_0 *is called a simple zero. In such case,* z_0 *is a pole of order n of the function* $\dfrac{1}{g(z)}$.

(2) *The poles of an analytic function interior to a simple closed contour are finite in number*

(3) *If a function f(z) is of the form* $f(z)=\dfrac{p(z)}{q(z)}$, $p(z)$ *and q(z) are analytic at* z_0, $p(z_0)\neq 0$, *then at the point* $z=z_0$, $f(z)=\dfrac{p(z)}{q(z)}$ *has a pole of order m if and only if q(z) has a zero of order m.*

(3) ***Essential Singularity:*** *A singularity which is not a pole, branch point or removable singularity is called an essential singularity. If z=a is an essential singularity of f(z), then the principal part of the Laurent series expansion has infinitely many terms. Equivalently,* $z= z_0$ *is an essential singularity if there does not exist any positive integer n such that* $\lim_{z\to z_0}(z-z_0)^n f(z) = A \neq 0$.

Example. *For* $f(z)=\cos\frac{1}{z}=1-\frac{1}{2!z^2}+\frac{1}{4!z^4}-\ldots$, $z = 0$ *is an essential singularity.*

Example. $f(z) = e^{\frac{1}{z-2}}$ *has an essential singularity at* $z = 2$.

Note: *If f(z) has an essential singularity. Then there exists at least one sequence* $\{z_n\}$ *such that* $\lim_{n\to\infty} f(z_n)=\infty$

Theorem 5.1.4.1. *If f is analytic in* $\Omega\setminus\{b\}$ *and bounded in* $0 < |z - b| < r$, *then either f is analytic at b or f has removable singularity at b.*

(4) ***Singularity at Infinity.*** *Under the inversion transformation* $w=\dfrac{1}{z}$, *the point* $z=\infty$ *in the extended complex plane corresponds to* $w=0$. *Thus, by letting* $z=\dfrac{1}{w}$ *in f(z), we obtain the function* $f\left(\dfrac{1}{w}\right)=F(w)$.

Then the nature of the singularity at $z = \infty$ [the point at infinity] of $f(z)$ is defined to be the same as that of $F(w)$ at $w = 0$.

Example. $f(z) = z^4$ has a pole of order 4 at $z = \infty$, since $F(w) = f\left(\frac{1}{w}\right) = \frac{1}{\omega^4}$ has a pole of order 4 at $w = 0$.

Similarly, $f(z) = \cos z$ has an essential singularity at $z = \infty$, since $F(w) = f\left(\frac{1}{w}\right) = \cos\frac{1}{w}$ has an essential singularity at $w = 0$.

Theorem 5.1.4.2. Limit point of zeros of a function is an isolated essential singularity of that function.

Example. Consider the function $\sin\frac{1}{z-a}$. Its zeros are given by $\sin\frac{1}{z-a} = 0$, i.e., $\frac{1}{z-a} = n\pi$

or $z = a + \frac{1}{n\pi}$ $(n = \pm 1, \pm 2, \ldots)$

Thus, we see that there is a sequence of zeros of $f(z)$, $\left\{a + \frac{1}{n\pi}\right\} n \in \mathbb{N}$ which has $z = a$ as its limit point.

Hence, $z = a$ is an isolated essential singularity of the function $\sin\frac{1}{z-a}$.

Theorem 5.1.4.3. Limit point of the poles of a function is a non-isolated essential singularity of that function

Example. Consider the function $f(z) = \dfrac{1}{\sin\frac{1}{z-a}}$.

Putting the denominator equal to zero, the poles of $f(z)$ are given by $\sin\frac{1}{z-a} = 0$,

i.e., $z = a + \frac{1}{n\pi}$ $(n = \pm 1, \pm 2, \ldots)$

Thus, we have a sequence of poles $a + \frac{1}{\pi}$, $a + \frac{1}{2\pi}$, $a + \frac{1}{3\pi}$, ...

The limit point of these poles is $z = a$.

Hence, $z = a$ is a non-isolated essential singularity of the given function.

Theorem. 5.1.4.4. (Casorati Weierstrass) Let f has an isolated essential singularity at z_0. If w is an arbitrary complex number, then $\exists$ points z in every nbd of z_0 such that $|f(z) - w|$ is arbitrary small, i.e., for positive number δ and $\in$, there exists z in $|z - z_0| < \delta$ such that $|f(z) - w| < \in$.

Proof: Let the above result be false, then there exist a complex number w_0 and a $\delta > 0$ satisfying $|f(z) - w_0| > \delta$, whenever $0 < |z - z_0| < \delta$. Thus $\left|\frac{f(z) - w_0}{z - z_0}\right| > \frac{\delta}{|z - z_0|} \to \infty$ as $z \to z_0$ and hence the function g defined by $g(z) = \frac{f(z) - w_0}{z - z_0}$ has a pole at $z = z_0$ and therefore $g(z) = (z - z_0)^{-m} h(z)$ for some suitable positive integer m and some analytic function h in the disk $|z - z_0| < \delta$. Thus $f(z) = w_0 + (z - z_0)^{-m+1} h(z)$, which

shows that the Laurent series expansion of f has only finitely many terms with negative exponents. This contradicts the hypothesis that $z = z_0$ is an essential singularity of f. Hence our assumption is incorrect. Thus the result is proved.

Theorem 5.1.4.5. *An analytic function f(z) is a polynomial of order n iff the only singularity of f(z) in extended complex plane is a pole of order n at ∞*

Theorem 5.1.4.6. *If f(z) is an entire function which has no singularity at infinity, then f(z) is constant.*

Proof: *As f(z) is analytic for all z∈ℂ, it can be extended in power series* $f(z)=a_0+a_1z+a_2z^2+\dots,$

$\therefore f\left(\frac{1}{z}\right)=a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\dots,$

$\Rightarrow$ *z=0 is an essential singularity of* $f\left(\frac{1}{z}\right)$

$\Rightarrow f\left(\frac{1}{z}\right)$ *is not analytic at z=0, which is not true.*

Hence, $a_n=0,\ n\geq 1$

Thus $f\left(\frac{1}{z}\right)=a_0 \Rightarrow f(z)=a_0$, *a constant.*

$\Rightarrow$ *f(z) is constant function*

Example 5.1.4.1. *Investigate the singularity of following functions at given points.*

(i) $\dfrac{\cot \pi z}{(z-a)^2}$ *at* $z=a,\ z=\infty$ *(ii)* $\sin\dfrac{1}{1-z}$ *at z=1* *(iii)* $\sin z-\cos z$ *at* $z=\infty$

(iv) $\operatorname{cosec}\dfrac{1}{z}$ *at z=0* *(v)* $\tan\dfrac{1}{z}$ *at z=0*

Solution:

(i) *Let* $f(z)=\dfrac{\cot \pi z}{(z-a)^2}=\dfrac{\cos \pi z}{\sin \pi z\,(z-a)^2}$

Poles of f(z) are obtained by equating to zero the denominator of f(z). Then, we have $(z-a)^2\sin\pi z=0$

$\Rightarrow \sin \pi z=0$ *or* $(z-a)^2=0$

Now, $\sin \pi z=0 \Rightarrow \pi z=n\pi$

$\Rightarrow z=n,$ *where n is any integer, and* $(z-a)^2=0 \Rightarrow z=a$

Hence, z=a is double pole and z = 0,±1, ±2,... are simple poles.

As $z=\infty$ *is a limit point of these simple poles, therefore* $z=\infty$ *is non-isolated essential singularity.*

(ii) *Let* $f(z)=\sin\dfrac{1}{1-z}$, *hence zeros of f(z) are given by* $\sin\dfrac{1}{1-z}=0$ *or* $\dfrac{1}{1-z}=n\pi$ *or* $z=1-\dfrac{1}{n\pi}$, *where n is any integer.*

$\therefore$ *z=1 is a limit point of these zeros, therefore z=1 is an isolated essential singularity.*

(iii) Let $f(z)=\sin z-\cos z$, hence zeros of $f(z)$ are given by $\sin z-\cos z=0 \Rightarrow \sin z=\cos z$

$\Rightarrow \tan z=1 \Rightarrow z=n\pi+\frac{\pi}{4}$, n is any integer. Thus, $z=\infty$ is a limit point of these zeros which is therefore an isolated essential singularity.

(iv) Let $f(z)=\operatorname{cosec}\frac{1}{z}=\frac{1}{\sin\left(\frac{1}{z}\right)}$, hence poles of $f(z)$ are given by $\sin\left(\frac{1}{z}\right)=0 \Rightarrow \frac{1}{z}=n\pi, z=\left(\frac{1}{n\pi}\right)$, where n is any integer. Since, $z=0$ is a limit point of these poles, therefore $z=0$ is a non-isolated essential singularity.

(v) Let $f(z)=\tan\left(\frac{1}{z}\right)=\frac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$

Hence, poles of $f(z)$ are given by $\cos\left(\frac{1}{z}\right)=0 \Rightarrow \frac{1}{z}=2n\pi\pm\frac{\pi}{2}$

$\Rightarrow z=\frac{1}{\left(2n\pm\frac{1}{2}\right)\pi}$, where n is any integer.

Since, $z=0$ is a limit point of these poles, therefore $z=0$ is a non-isolated essential singularity.

Example 5.1.4.2 Find the kind of the singularities of the following functions.

(i) $\frac{1-e^z}{1+e^z}$ at $z=\infty$ (ii) $\frac{1}{\sin z-\cos z}$ at $z=\frac{\pi}{4}$

(iii) $\sin\frac{1}{z}$ at $z=0$ (iv) $z \operatorname{cosec} z$ at $z=\infty$.

Solution:

(i) Let $f(z)=\frac{1-e^z}{1+e^z}$, Poles of $f(z)$ are obtained by equating to zero the denominator of $f(z)$.

$\therefore 1+e^z=0 \Rightarrow e^z=-1=e^{2n\pi i+\pi i}$

$\Rightarrow z=(2n+1)\pi i$, where n is any integer.

Hence, $z=(2n+1)\pi i$, $n \in \mathbb{Z}$ are the simple poles of $f(z)$ and $z=\infty$ is a limit point of these poles. Therefore $z=\infty$ is a non-isolated essential singularity.

(ii) Let $f(z)=\frac{1}{\sin z-\cos z}$, hence poles of $f(z)$ are given by $\sin z-\cos z=0 \Rightarrow \tan z=1$

$\Rightarrow z=n\pi+\frac{\pi}{4}$, n is any integer. At $n=0$, $z=\frac{\pi}{4}$ which is a simple pole.

(iii) *Let* $f(z)=\sin\frac{1}{z}$, *hence zeros of f(z) are given by* $\sin\frac{1}{z}=0$ *or* $\frac{1}{z}=n\pi$ *or* $z=\frac{1}{n\pi}$, *n. is any integer. Obviously, z=0 is a limit point of zeros of f(z), hence z=0 is an isolated essential singularity of f(z).*

(iv) *Let* $f(z)=z\operatorname{cosec} z=\frac{z}{\sin z}$, *hence poles of f(z) are given by* $\sin z=0 \Rightarrow z=n\pi, n$ *is any integer*
Since, $z=\infty$ *is a limit point of these poles, therefore* $z=\infty$ *is a non -isolated essential singularity.*

5.2. ENTIRE FUNCTION

A function which is analytic everywhere in the entire complex plane [i.e., everywhere except at ∞] is called an entire function or integral function. The functions e^z, *sin z, cos z, sinhz, coshz are entire functions. An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely, if a power series has an infinite radius of convergence, then it represents an entire function. Note that by Liouville's theorem a function which is analytic everywhere including ∞ must be a constant. Thus, a function having no singularity anywhere in the extended complex plane is a constant.*

5.3. MEROMORPHIC FUNCTION

A function which is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function or in other words singularities of a meromorphic function are at most poles. Thus on the extended complex plane, a meromorphic function can have essential singularity at infinity also.

Example. $f(z)=\frac{z+1}{(z+2)(z-1)^2}$, *is analytic everywhere in the finite plane except at the poles; z=-2 (simple pole) and z = 1 (pole of order two), so f(z) is a meromorphic function.*

Note: *Every analytic function is meromorphic but every meromorphic function need not be analytic.*

5.4. RESIDUE

Let f(z) be single-valued and analytic inside and on a circle C except at the point z = a chosen as the centre of C. Then the integral $\frac{1}{2\pi i}\oint_C f(z)dz$ *is called the residue of f(z) at z=a.*

Let f(z) has a Laurent series about z = a given by $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-a)^n$

$$= a_0+a_1(z-a)+a_2(z-a)^2+\ldots+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\ldots \qquad \ldots(1)$$

$$\text{where } a_n=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz,\ n=0,\pm 1,\pm 2,\ldots \qquad \ldots(2)$$

In the special case n = -1, we have from (2)

$$\oint_C f(z)\,dz = 2\pi i a_{-1}$$

$$a_{-1}=\frac{1}{2\pi i}\oint_C f(z)dz$$

Thus the residue of f(z) at z=a is the coefficient of $(z-a)^{-1}$ in the Laurent Series expansion of f(z) around z=a.

...(3)

5.4.1. Calculation of Residue at finite singularities:

Let z = a be a pole of f(z) order m, then the residue of f(z) at z = a is given by

$$\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

if m = 1, i.e., if z = a is simple pole of f(z), then

$$\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}(z-a)f(z) \quad [\because 0! = 1]$$

Example. *Consider* $f(z)=\dfrac{z+1}{(z-2)^2(z-1)}$

Then, f(z) has poles at z = 2 of order 2 and at z = 1 of order 1.

$$\text{Residue at } z=2=\lim_{z\to 2}\frac{1}{1!}\frac{d}{dz}\left\{(z-2)^2\left(\frac{z+1}{(z-2)^2(z-1)}\right)\right\}=\lim_{z\to 2}\frac{d}{dz}\left(\frac{z+1}{(z-1)}\right)=-2$$

$$\text{Residue at } z=1=\lim_{z\to 1}(z-1).\left(\frac{z+1}{(z-2)^2(z-1)}\right)=2$$

Note: *If z = a is an essential singularity of f(z), then the residue of f(z) at z = a can be found by using any known series expansion.*

Example. *If* $f(z)=\sin\frac{1}{z}$, *then z = 0 is an essential singularity, as* $\sin\frac{1}{z}=\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}\cdots$ ...(1)

then $\operatorname*{Res}_{z=0} f(z)=$ *coefficient of* $\frac{1}{z}$ *in (1) = 1*

Example 5.4.1.1. *Let the only singularities of f(z) be poles of order 2 and 1 at z = 1 and z = 2 with residue 1 and 2 respectively. If f(0) = 2,* $f(3)=\frac{17}{4}$, *then find the function f(z) and find its Laurent series expansion in 1 < |z| < 2.*

Solution: *Given f(z) has*

(a) *pole of order 2 at z = 1 with residue 1*

(b) *pole of order 1 at z = 2 with residue 2*

Also the only singularities of f(z) are poles, hence f(z) can be expressed as

$$f(z)=a_0+\sum_{n=1}^{\infty}a_n z_n+\frac{1}{(z-1)}+\frac{b}{(z-1)^2}+\frac{2}{(z-2)} \qquad ...(1)$$

Now, as f(z) has no pole at z = ∞,

$\Rightarrow f\left(\frac{1}{z}\right)$ *has no pole at z = 0.*

$\Rightarrow$ *Principal part of* $f\left(\frac{1}{z}\right)\left(i.e., \sum_{n=1}^{\infty} a_n z^{-n}\right)$ *contains no term*

$\Rightarrow a_n = 0$ *for n = 1, 2, 3, ...*

Hence (1) becomes, $f(z) = a_0 + \frac{1}{(z-1)} + \frac{b}{(z-1)^2} + \frac{2}{(z-2)}$

Now, $f(0) = 2 = a_0 - 1 + b - 1$

$f(3) = \frac{17}{4} = a_0 + \frac{1}{2} + \frac{b}{4} + 2 \Rightarrow a_0 = 1$ *and* $b = 3$

Hence $f(z) = 1 + \frac{1}{(z-1)} + \frac{3}{(z-1)^2} + \frac{2}{(z-2)}$

Now, to find Laurent series expansion in $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1, \left|\frac{z}{2}\right| < 1$

$$f(z) = 1 + \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{z^2\left(1-\frac{1}{z}\right)^2} - \frac{1}{\left(1-\frac{z}{2}\right)} = 1 + \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{3}{z^2}\left[1-\frac{1}{z}\right]^{-2} - \left[1-\frac{z}{2}\right]^{-1}$$

$$= 1 + \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right] + \frac{3}{z^2}\left[1 + \frac{2}{z} + \frac{2.3}{2!}\left(\frac{1}{z}\right)^2 + \dots\right] - \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots\right]$$

$$= 1 + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{3}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n (n+1) - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = 1 + \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}\left[1 + \frac{3(n+1)}{z}\right] - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

Example 5.4.1.2. *The function f(z) has a double pole at z = 1 with residue 2, a simple pole at z=0 with residue 3, is analytic at all other finite points of the plane and is bounded as* $|z| \to \infty$ *. If f(3)=5, f(5)=13/5, find f(z).*

Solution: *Res (z = 1) = 2, (order 2) and Res(z = 0) = 3, (order 1). Hence f(z) will be expressible as*

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \frac{2}{z-1} + \frac{b}{(z-1)^2} + \frac{3}{z} \qquad ...(1)$$

Since, it is given that f(z) is bounded as $|z| \to \infty$*, so let* $|f(z)| \le M$ *(M>0). Therefore, f(z) has no singularity at z=* ∞*, thus f(w) has no singularity at w=0, where* $w = \frac{1}{z}$*. Hence, the principal part of f(w) contains no term so that* $a_n = 0\ \forall n$*.*

$\left(\sum_{1}^{\infty} a_n w^{-n}\right.$ *is the principal part in the Laurent series expansion of f(w)*$\left.\right)$*.*

Now (1) becomes $f(z) = a_0 + \frac{2}{z-1} + \frac{b}{(z-1)^2} + \frac{3}{z} \qquad ...(2)$

$\Rightarrow$ Now $f(3) = 5 = a_0+1+\frac{b}{4}+1=5$ and $f(5)= a_0 + 1/2 + b/16+3/5=13/5$ $\Rightarrow a_0= 1, b = 8.$

$$\Rightarrow f(z) = 1 + \frac{2}{z-1} + \frac{8}{(z-1)^2} + \frac{3}{z}$$

5.4.2. Residue at Infinity:

A function $f(z)$ is said to be analytic at $z=\infty$, if the function $f\left(\frac{1}{z}\right)$ is analytic at the origin. Similarly, we say that $f(z)$ has a zero of order m, pole of order n and an isolated singularity at ∞, if $f\left(\frac{1}{z}\right)$ has a zero of order n, pole of order m and isolated singularity at origin respectively.

Result: *If $f(z)$ has an isolated singularity at infinity or is regular there and if C is a large circle which encloses all the finite singularities of $f(z)$, then the residue at $z = \infty$ is defined to be $\frac{1}{2\pi i}\int_C f(z)dz$ in clockwise direction taken round C in the negative sense (negative w.r.t. origin) provided that the integral has a definite value. Taking the integral along C in the anticlockwise direction (in positive sense)*

Thus residue of $f(z)$ at $z = \infty$ is $= -\frac{1}{2\pi i}\int_C f(z)dz$

$=$ *negative of coefficient of $\left(\frac{1}{z}\right)$ in the expansion of $f(z)$ in the neighbourhood of $z = \infty$.*

An Important Observation: *The residue at a finite point is zero if the function is analytic at that point or have removable singularity at that point but may not be so at infinity.*

For Example. *If $f(z) = \frac{1}{z-a}$, then f is analytic at $z = \infty$ but residue of $f(z)$ at $z = \infty$ is given by*

$$-\frac{1}{2\pi i}\int_C f(z)dz = -\frac{1}{2\pi i}\int_0^{2\pi}\frac{rie^{i\theta}d\theta}{r.e^{i\theta}} = -1.$$

5.4.3 Working Rules for Computing the Residue:

1. $\text{Res}(z=a) = \lim_{z\to a}(z-a)f(z)$ *for simple pole.*

2. $\text{Res}(z=a) = \frac{\phi^{(m-1)}(a)}{(m-1)!}$ *for pole of order m, if* $f(z) = \frac{\phi(z)}{(z-a)^m}$

3. $\text{Res}(z=a) = \frac{1}{2\pi i}\int_C f(z)dz$ *for pole of any order. (If 'a' is the only singularity inside C)*

4. $\text{Res}(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)dz = -\text{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)$

5. $\text{Res}(z=\infty)$ = *negative of the coefficient of* $\left(\frac{1}{z}\right)$ *in the expansion of f(z) in the neighbourhood of* $z=\infty$.

6. *If* $f(z)=\frac{\phi(z)}{\psi(z)}$ *has a simple pole at z=a, then* $\text{Res}\ (z=a)=\frac{\phi(a)}{\psi'(a)}$.

7. *Let* $P(z)$ *and* $Q(z)$ *be complex polynomials. If* $\deg(Q(z))\geq \deg(P(z))+2$, *then* $\oint_C \frac{P(z)}{Q(z)}dz=0$, *where* *C encloses all singular points of* $\frac{P(z)}{Q(z)}$

5.4.4. Some Theorems:

Theorem 5.4.4.1. *If f(z) has a simple pole at* z_0 *and g(z) is analytic at* z_0 *such that* $g(z_0)\neq 0$, *then* $\text{Res}_{z=z_0} f(z)g(z)=g(z_0)\text{Res}_{z=z_0} f(z)$.

Theorem 5.4.4.2. *If* $f(z)=(z-a)^{-n}(z-b)^{-m}$, *where m and n are positive integers, then* $\text{Res}_{z=a} f(z)=-\text{Res}_{z=b} f(z)$

Theorem 5.4.4.3. *If function f(z) has an isolated singularity at* z_0 *and f(z) is even in* $z-z_0$ *,i.e.,* $f(z-z_0)=f(-(z-z_0))$, *then* $\text{Res}_{z=z_0} f(z)=0$

Theorem 5.4.4.4. *Suppose* ϕ *is analytic at* z_0 *and* $\phi(z_0)\neq 0$. *If a function g has a zero of order three at* z_0,

then $\text{Res}_{z=z_0}\frac{\phi(z)}{g(z)}=3\left\{\frac{-\phi''(z_0)}{g'''(z_0)}-\frac{1}{2}\frac{\phi'(z_0)g^{(iv)}z_0}{[g'''(z_0)]^2}\right\}$

Theorem:5.4.4.5. *If n is an even integer, then all the singularities of* $\tan^{n-1}(\pi z)$ *are* $z=\left(n+\frac{1}{2}\right)$ *and residue at* $z=\frac{1}{2}$ *is* $\frac{(-1)^{n/2}}{\pi}, n\in\mathbb{Z}$

Theorem 5.4.4.6. *If f(z) has only finite singularities and* $\lim_{z\to\infty} zf(z)=0$, *then there always exist R>0 such that* $\{z^2f(z):|z|>R\}$ *is bounded.*

Theorem 5.4.4.7. *Let f(z) be an entire function such that* $\lim_{z\to\infty}\left|\frac{f(z)}{z}\right|=0$, *then f(z) is a constant. If* $\lim_{z\to\infty}\left|\frac{f(z)}{z^n}\right|=0$ *then f must be a polynomial of degree* $\leq n-1$

Theorem 5.4.4.8. *For fixed positive integers n and m,* $\int_{|z|=n} z^m \tan z\,dz = -\sum_{k=-n}^{n}\left(k+\frac{1}{2}\right)^m$

Theorem 5.4.4.9. *If f(z) be a meromorphic function in* $\mathbb{C}$ *and* $\exists$ *a positive integer n and M, R > 0 such that* $|f(z)| \le M|z|^n$ *for* $|z| \ge R$, *then f(z) is a rational function.*

Theorem 5.4.4.10. *Let f(z) be analytic at* z_0 *such that* $f(z_0) \neq 0$ *and g(z) has a zero of order two at* z_0, *then*

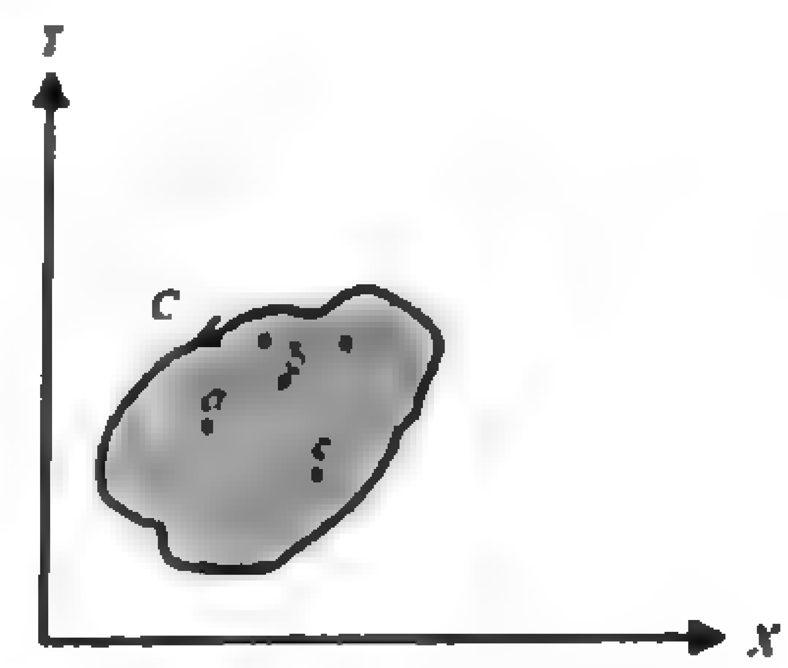
$$\operatorname{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3[g''(z_0)]^2}$$

Theorem: 5.4.4.11. *Suppose f(z) and g(z) are both analytic at* z_0 *and* $f(z_0) \neq 0$ *and* z_0 *is a first order zero for g, then*

$$\operatorname{Res}_{z=z_0} \frac{f(z)}{[g(z)]^2} = \frac{f'(z_0)g'(z_0) - f(z_0)g''(z_0)}{[g'(z_0)]^3}$$

5.5. THE RESIDUE THEOREM

Let f(z) be single-valued and analytic inside and on a simple closed curve C except at the singularities $z_1, z_2, z_3, \ldots z_n$ *inside C which have residues given by* $a_{-1}, a_{-2}, a_{-3}, \ldots$. *Then the residue theorem states that* $\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+a_{-2}+a_{-3}+\ldots)$, *i.e., the integral of f(z) around C is* $2\pi i$ *times the sum of the residues of f(z) at the singularities enclosed in C. Thus* $\oint_C f(z)\,dz = 2\pi i \sum_{m=1}^{n} \operatorname{Res}_{z=z_m} f(z)$



Result: *If an analytic function has singularities at a finite number of points (including that at infinity), then the sum of the residues at these points along with infinity is zero.*

Exercise. 5.5.1. *The residue at* $z=\infty$ *of* $f(z) = \dfrac{z^3}{(z-2)(z-3)(z-5)}$ *is*

(a) 0 *(b) -10* *(c) 10* *(d) none of these*

Solution: *Residue at* $(z=\infty)$

$$= -\{\operatorname{Res}(z=2) + \operatorname{Res}(z=3) + \operatorname{Res}(z=5)\}$$

$$= -\left\{(z-2)\frac{z^3}{(z-2)(z-3)(z-5)}\bigg|_{z=2} + (z-3)\frac{z^3}{(z-2)(z-3)(z-5)}\bigg|_{z=3} + (z-5)\frac{z^3}{(z-2)(z-3)(z-5)}\bigg|_{z=5}\right\}$$

$$= -\left\{\frac{z^3}{(z-3)(z-5)}\bigg|_{z=2} + \frac{z^3}{(z-2)(z-5)}\bigg|_{z=3} + \frac{z^3}{(z-2)(z-3)}\bigg|_{z=5}\right\}$$

$$= -\left\{\frac{8}{(-1)(-3)} + \frac{27}{-2} + \frac{125}{3\times 2}\right\} = -\left\{\frac{8}{3} - \frac{27}{2} + \frac{125}{6}\right\} = -\left\{\frac{16-81+125}{6}\right\} = -\left\{\frac{141-81}{6}\right\} = -10$$

5.6. *Characterization of Rational Functions*

If a single-valued function f(z) has no singularities other than pole in the extended complex plane is a rational function.

Note: (1) *A function f(z) is rational iff it is meromorphic in the extended complex plane.*

(2) *Let f(z) be meromorphic in $\mathbb{C}$ and there exist a natural number n, $M > 0$ and $R > 0$ such that $|f(z)| \le M|z|^n$ for $|z| > R$. Then f is a rational function.*

5.7. *Characterization of Polynomials*

A function which has no singularity in the extended complex plane other than a pole of order n at infinity is a polynomial of degree n.

Theorem: *In entire function f(z) whose singularity at infinity is at the most, a pole is necessarily a polynomial*

Theorem: *The order of a zero of a polynomial equals the order of its first non-vanishing derivative.*

Theorem: *If a function f(z) is analytic for all finite values of z and as $|z| \to \infty$, $|f(z)| = A|z|^K$, then f(z) is a polynomial of degree $\le K$.*

PRACTICE SET – I

Exercise 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic except for a simple pole at $z = 0$ and let $g : \mathbb{C} \to \mathbb{C}$ be analytic. Then, the value of $\dfrac{\operatorname{Res}_{z=0}\{f(z)g(z)\}}{\operatorname{Res}_{z=0} f(z)}$ is* **(GATE-2011)**

(a) $g(0)$ (b) $g'(0)$ (c) $\lim_{z\to 0} zf(z)$ (d) $\lim_{z\to 0} zf(z)g(z)$

Exercise 2. *Let $f(z) = \dfrac{z}{8-z^3}, z = x+iy$, then $\operatorname{Res}_{z=2} f(z)$ is* **(GATE-2011)**

(a) -1/8 (b) 1/8 (c) -1/6 (d) 1/6

Exercise 3. *Let $\gamma=\{z\in \mathbb{C} : |z| = 2\}$ be oriented in the counter-clockwise direction. Let $I = \dfrac{1}{2\pi i}\oint_\gamma z^7 \cos\left(\dfrac{1}{z^2}\right)dz$. Then, the value of I is equal to ____________* **(GATE-2016)**

Exercise 4. Let $f(z) = \dfrac{z-1}{\exp\left(\dfrac{2\pi i}{z}\right)-1}$. Then (CSIR UGC NET DEC-2013)

(a) f has an isolated singularity at $z=0$
(b) f has a removable singularity at $z=1$
(c) f has infinitely many poles
(d) each pole of f is of order 1.

Exercise 5. For the function $f(z) = \dfrac{z-\sin z}{z^3}$, at the point $z = 0$ is a/an

(a) pole of order 3
(b) pole of order 2
(c) essential singularity
(d) removable singularity

Exercise 6. $f(z) = \log(z^2 + z - 2)$ has branch points at

(a) $z=1$ (b) $z = 2$ (c) $z = -2$ (d) $z=-1$

Exercise 7. An example of function with a non isolated essential singularity at $z=2$ is

(a) $\tan\dfrac{1}{z-2}$ (b) $\sin\dfrac{1}{z-2}$ (c) $e^{-(z-2)}$ (d) $\tan\dfrac{z-2}{z}$

5.8. THE ARGUMENT THEOREM

Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Then $\dfrac{1}{2\pi i}\oint_C \dfrac{f'(z)}{f(z)}dz = N - P$, where $N = \sum_{i=1}^{k} n_i$; N is the sum of multiplicities of zeros of $f(z)$ lying inside the region. n_i is the multiplicity of ith zero of $f(z)$, $P = \sum_{j=1}^{l} m_j$; P is the sum of the multiplicities of poles of $f(z)$ lying within the region and m_j is the order of jth pole inside the region.

Example 5.8.1. Evaluate the integral $\oint_C \dfrac{f'(z)}{f(z)}dz$ when $f(z) = \dfrac{(z^2-z-2)^3}{(z^2+2)^2}$ and C is the circle $|z| = 3.5$ taken in the positive sense.

Solution. Given $f(z) = \dfrac{(z^2-z-2)^3}{(z^2+2)^2}$

Zeros of $f(z)$ are given by $z^2 - z - 2 = 0 \Rightarrow z = -1, 2$.
Clearly $z = -1$, $z = 2$ lies inside $|z| = 3.5$
Hence $N = 3 + 3 = 6$
Poles of $f(z)$ are given by $z^2 + 2 = 0 \Rightarrow z = \pm 2i$
Also, $z = \pm 2i$ lies inside $|z| = 3.5$
Hence $P = 2 + 2 = 4$

$$\therefore \oint_C \frac{f'(z)}{f(z)}dz = 2\pi i[N-P] = 2\pi i[6-4] = 4\pi i.$$

Applications:

(1) Let $f(z)=a_0+a_1z+a_2z^2+...+a_nz^n$, where $a_n \neq 0$ and $a_0, a_1, ..., a_n$ are complex constants. Then

(a) $\frac{1}{2\pi i}\int_\gamma \frac{z f'(z)}{f(z)}dz = -\frac{a_{n-1}}{a_n}$;

(b) $\frac{1}{2\pi i}\int_\gamma \frac{z^2 f'(z)}{f(z)}dz = \frac{a_{n-1}^2 - 2a_n a_{n-2}}{a_n^2}$, where γ is a simple closed contour enclosing all zeros of P(z).

(2) If f(z) and g(z) be analytic inside and on a simple closed curve C such that f(z) has zeros at $n_1, n_2, ..., n_p$ and poles at $m_1, m_2, ..., m_q$ of orders (multiplicities) $r_1, r_2, ..., r_p$ and $s_1, s_2, ..., s_q$ respectively, then

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)}dz = \sum_{t=1}^{p} r_t g(n_t) - \sum_{t=1}^{q} q_t g(m_t)$$

5.9. IMPORTANT THEOREMS

5.9.1. Picard's Theorem: *Every non-constant entire function omits atmost one complex number as its value. In other words, if an entire function omits two values, then it is constant.*

5.9.2. Gauss Theorem: *If zeros of f(z) lies in a set S contained in a domain D, then zeros of $f'(z)$ will lie on convex hull of S. Open disc and Closed disc are convex sets.*

5.9.3. Luca's Theorem: *If zeros of f(z) lies on upper half complex plane, then zeros of $f'(z)$ also lies on that same upper half plane.*

5.9.4. Riemann's Theorem: *Let $|f(z)| \leq M\ \forall z$ such that $0<|z-z_o|<\delta$ and f is analytic in $0<|z-z_o|<\delta$*

Define $g(z)=\begin{cases}(z-z_o)^2 f(z), & z\neq z_o \\ 0, & z=z_o\end{cases}$

Then, g(z) is analytic in the disc.

5.9.5. Argument Principle: *Let a function f(z) be analytic within and on a closed contour C, having N zeros inside C but no zero on C. Then $N=\frac{1}{2\pi}\Delta_c \arg f(z)$, where Δ_c arg f(z) denotes the change in argf(z), as z moves along C in the positive sense.*

Example 5.9.5.1. *Consider the equation z^4+z^3+1*

(i) *Prove that the equation does not have any real root.*

Proof: *Since the coefficients of the equation are all real and positive and so it will not be satisfied by any positive real value of z. Hence, the equation does not have any real positive root.*

$f(-x)=x^4-x^3+1=x^3(x-1)+1>0$, if $x>1$ ...(1)

and $f(-x)=x^4+(1-x^3)=x^4+(1-x)(1+x^2+x)>0$ if $0<x<1$. ...(2)

Thus f(-x) > 0 if x > 1.

f(-x) > 0 if 0 < x < 1

(1) and (2) cannot hold simultaneously. Hence, the equation does not have any negative real root. Hence the result (i) follows.

(ii) *Prove that the equation does not have any purely imaginary root.*

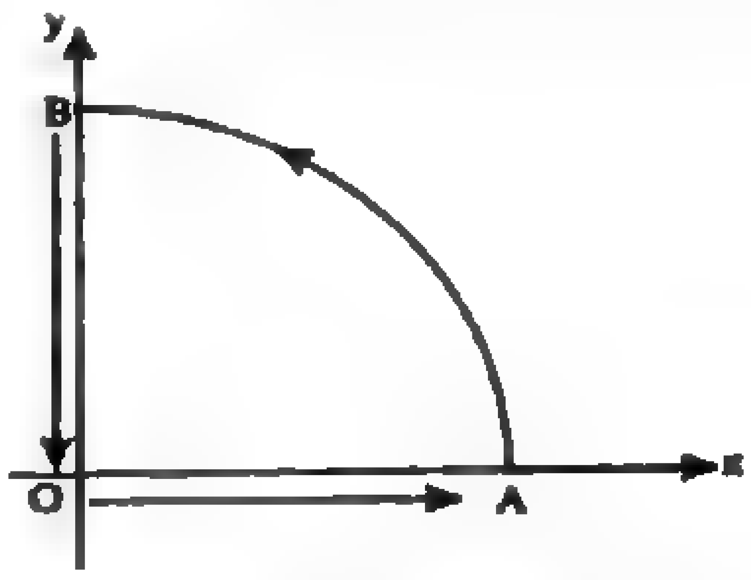
Proof: $f(iy)=(iy)^4+(iy)^3+1 \Rightarrow y^4-iy^3+1=0$

Thus $y^4+1=0, y^3=0$ *(equating real and imaginary parts)*

These two equations cannot hold simultaneously and hence the result (ii) follows.

(iii) *Determine the number of roots in the first quadrant.*

Proof: *For this let* $z=Re^{i\theta}, 0\le\theta\le\pi/2, R\to\infty$ *define the first quadrant OABO. Let C denote the complete boundary of this quadrant. Since f(z) is a polynomial and so it is analytic* $\forall z$ *except at* $z=\infty$. *Hence, it is analytic within and upon C.*



(a) ***Along OA:*** *On this line z=x and x varies from 0 to* ∞. *Also* $u+iv=f(x)=x^4+x^3+1$.

$$\Rightarrow u=x^4+x^3+1, v=0$$

$$\Rightarrow \arg f(x)=\tan^{-1}\frac{v}{u}$$

$$=\tan^{-1}\frac{0}{x^4+x^3+1}=\tan^{-1}0$$

$$\Rightarrow \arg f(x)=0 \quad \forall x\ge 0 \quad \Rightarrow \Delta_{OA}\arg f(x)=0.$$

(b) ***Along AB:*** *z=* $Re^{i\theta}$ *where θ varies from 0 to π/2. Thus* $f(z)=R^4e^{i4\theta}+R^3e^{i3\theta}+1$

$$\Rightarrow f(z)=R^4e^{i4\theta}\left[1+\frac{1}{Re^{i\theta}}+\frac{1}{R^4e^{i4\theta}}\right]\to R^4e^{i4\theta} \text{ as } R\to\infty$$

$$\arg f=\tan^{-1}\frac{v}{u}=\tan^{-1}\left(\frac{R^4\sin 4\theta}{R^4\cos 4\theta}\right)=4\theta$$

$$\Delta_{AB}\arg f(z)=[4\theta]_0^{\pi/2}=4\left(\frac{\pi}{2}-0\right)=2\pi$$

(c) ***Along BO:*** *On this line z= i y and y varies from* ∞ *to 0.*

$$f(z)=u+iv=(iy)^4+(iy)^3+1=y^4+1+i(-y^3)$$

$$\arg f=\tan^{-1}\frac{v}{u}=\tan^{-1}\left(\frac{-y^3}{y^4+1}\right)=-\tan^{-1}\frac{y^3}{y^4+1}$$

$$\Delta_{BO}\arg f=-\left[\tan^{-1}\frac{y^3}{y^4+1}\right]_{\infty}^{0}=-[\tan^{-1}0-\tan^{-1}0]=0$$

$$\Rightarrow \Delta_C\arg f(z)=\Delta_{OA}\arg f+\Delta_{AB}\arg f+\Delta_{BO}\arg f=0+2\pi+0=2\pi \Rightarrow N=\frac{1}{2\pi}\Delta_C\arg f=\frac{2\pi}{2\pi}=1$$

Hence, by argument principle the equation has one complex root in the first quadrant.

5.9.6. Rouche's Theorem: *If f(z) and g(z) are analytic inside and on a simple closed curve C and if $|g(z)|<|f(z)|$ on C, then f(z) + g(z) and f(z) have the same number of zeros inside C.*

Example 5.9.6.1. *Using Rouche theorem, find the number of roots of z^7-4z^3+z-1, which lie interior to the unit circle $|z|=1$.*

Solution: *Here, take $f(z)=-4z^3$ and $g(z)=z^7+z-1$. Now,* $\left|\frac{g(z)}{f(z)}\right|=\frac{|z^7+z-1|}{|-4z^3|}\le\frac{|z|^7+|z|+1}{4|z|^3}=\frac{3}{4}<1$ *on* $|z|=1$.

Thus conditions of Rouche's theorem are satisfied. So, $f(z)=-4z^3$ and $f(z)+g(z)=z^7-4z^3+z-1$ have the same number of zeros inside $|z|=1$. But f(z) has a zero of order 3 at the origin. Hence, we conclude that z^7-4z^3+z-1 has 3 roots inside $|z|=1$.

5.9.7. Identity Theorem: *Suppose that f is analytic in a domain D. If the set of zeros of f has a limit point in D, then $f(z)\equiv 0$ in D.*

Corollary: *Suppose that f and g are analytic in a domain D. If set of zeros of f-g has a limit point in D, then $f(z)\equiv g(z)\ \forall z\in D$.*

Result: *If two functions agree on a connected domain, then they agree for all z, i.e., if $f(z)=g(z)\ \forall z\in D$, where D is connected domain, then $g(z)=f(z)\ \forall z\in\mathbb{C}$*

5.9.8. Open Mapping Theorem: *In complex analysis, the open mapping theorem states that if U is a connected open subset of the complex plane $\mathbb{C}$ and $f:U\to\mathbb{C}$ is a non-constant holomorphic function, then f is an open map (i.e., if maps open subsets of U to open subsets of $\mathbb{C}$).*

5.9.9. Schwarz Lemma Statement: *Let $D=\{z:|z|<1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $f:D\to\overline{D}$ be an analytic function with f(0)=0. The Schwarz lemma states that if $|f(z)|\le|z|$ for all $z\in D$ and $|f'(0)|\le 1$, and if the equality $|f(z)|=|z|$ holds for any $z\neq 0$ or $|f'(0)|=1$, then f is a rotation, i.e., $f(z)=az$ with $|a|=1$.*

Extension of Schwarz Lemma: *If a function f(z) is such that*

(i) $f(z)$ is analytic in $|z|<1$ and

(ii) $|f(z)|\le 1$ for all $z, |z|<1$

(iii) $f(z)$ has zero of order m at $z=0$. Then,

(i) $|f(z)|\le |z|^m \ \forall |z|<1$

(ii) $|f^m(0)|\le m$

Corollary: Schwarz Pick Theorem: *If f(z) is an analytic function in $|z|<1$, then*

(i) $|f(z)|\le \dfrac{|f(0)|+|z|}{1+|f(0)||z|}$ (ii) $|f'(z)|\le \dfrac{1-|f(z)|^2}{1-|z|^2}$

5.9.10. Mittag Leffler's Expansion Theorem: *Suppose that the only singularities of f(z) in the finite z plane are the simple poles $a_1, a_2, a_3, \ldots$ arranged in order of increasing absolute value, i.e., $0<|a_1|<|a_2|\ldots<|a_n|$ with residues $b_1, b_2, b_3, \ldots$ respectively. Let C_N be circle of radius R_N which do not pass through any poles and $|f(z)|<M$, on C_N where M is independent of N and $R_N\to\infty$ as $N\to\infty$*

Then, Mittag Leffler's expansion theorem states that $f(z)=f(0)+\sum_{n=1}^{\infty} b_n\left\{\dfrac{1}{z-a_n}+\dfrac{1}{a_n}\right\}$

Example 5.9.10.1. *Prove that* $\cot z=\dfrac{1}{z}+\sum_n\left(\dfrac{1}{z-n\pi}+\dfrac{1}{n\pi}\right)$, *where the summation extends over* $n=\pm1,\pm2,\ldots$

Solution: *Consider the function* $f(z)=\cot z-\dfrac{1}{z}=\dfrac{z\cos z-\sin z}{z\sin z}$.

Then f(z) has simple poles at $z=n\pi, n=\pm1,\pm2,\pm3,\ldots$ *and the residue at these poles is given by*

$$\operatorname*{Res}_{z=n\pi} f(z)=\lim_{z\to n\pi}(z-n\pi)\left(\frac{z\cos z-\sin z}{z\sin z}\right)=\lim_{z\to n\pi}\left(\frac{z-n\pi}{\sin z}\right)\lim_{z\to n\pi}\left(\frac{z\cos z-\sin z}{z}\right)=1$$

Also, at z = 0, f(z) has a removable singularity

and, $\lim_{z\to0}\left(\cot z-\dfrac{1}{z}\right)=\lim_{z\to0}\left(\dfrac{z\cos z-\sin z}{z\sin z}\right)=0$ *[by L'Hospital rule]*

$\therefore f(0)=0$.

Now, let C_N be a circle of radius $R_N=\left(N+\dfrac{1}{2}\right)\pi$

Clearly, C_N encloses all the singularities of f(z) and $R_N\to\infty$ as $N\to\infty$.

Now, to prove $f(z)\le M\ \forall z\in C_N$ where M is independent of N.

Clearly, $\left|\dfrac{1}{z}\right|<1\ \forall |z|\ge1$. *Now, taking $z=x+iy$,* $|\cot z|^2=1+\dfrac{\cos 2x}{\sin^2 x+\sinh^2 y}$

Also $\sin R=\pm 1$ $\left[\because R=\left(N+\dfrac{1}{2}\right)\pi\right]$

Now when, $z \in C_N$ *and* $|x| \geq R - \frac{\pi}{4}$, *then* $\cos(2x) \leq 0 \Rightarrow |\cot z|^2 \leq 1$ *when* $|x| \geq R - \frac{\pi}{4}, z \in C_N$

Also if $z \in C_N$ *and* $|x| \leq R - \frac{\pi}{4}$, *then* $\sinh^2 y \geq y^2 = R^2 - x^2 \geq \frac{R\pi}{2} - \frac{\pi^2}{16} \geq \frac{3\pi^2}{16} > 1$

Thus, $|\cot z|^2 \leq 2$ *when* $|x| \leq R - \frac{\pi}{4}, z \in C_N$

Hence $|\cot z|^2 \leq 2 \; \forall z \in C_N \; \therefore |f(z)| = \left|\cot z - \frac{1}{z}\right| \leq 3 \; \forall z \in C_N$

$\Rightarrow |f(z)| \leq M \;\; \forall z \in C_N$, *where* $M = 3$ *is independent of* N.
Thus all conditions of Mittag Leffler's Theorem are satisfied and therefore

$$f(z) = f(0) + \sum_{n=1}^{\infty} 1\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) + \sum_{n=-1}^{-\infty}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$$

$$\Rightarrow \cot z - \frac{1}{z} = 0 + \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) + \sum_{n=-1}^{-\infty}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$$

$$\Rightarrow \cot z = \frac{1}{z} + \sum_{n}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right),$$ *where the summation is taken over* $n = \pm 1, \pm 2, \ldots$

PRACTICE SET – II

Exercise 1. *If* $a > e$, *then prove that the equation.* $e^z = az^n$ *has n roots inside the circle* $|z| = 1$.

Exercise 2. *Prove that all the roots of* $z^7 - 5z^3 + 12 = 0$ *lie between the circle* $|z| = 1$ *and* $|z| = 2$.

Exercise 3. *Show that* $z^5 + 15z + 1 = 0$ *has one root in the disc* $|z| < \frac{3}{2}$ *and four roots in annulus* $\frac{3}{2} < |z| < 2$.

Exercise 4. *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$ *be the unit disc. Let* $f : D \to \mathbb{C}$ *be an analytic function satisfying* $f\left(\frac{1}{n}\right) = \frac{2n}{3n+1}$ *for* $n \geq 1$. *Then* **(CSIR UGC NET JUNE-2011)**

(a) $f(0) = 2/3$
(b) f *has a simple pole at* $z = -3$
(c) $f(3) = 1/3$
(d) no such f *exists*

Exercise 5. *If* $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ *and C is the circle* $|z| = 5$, *then* $\int_C \frac{zf'(z)}{f(z)} dz =$

(a) $2\pi i$ *(b)* 2 *(c)* $4\pi i$ *(d)* πi

KEY POINTS

- *The point $z=z_o$ is called an isolated singularity of f(z) if there exists $\delta>0$ such that circle $|z-z_o|=\delta$ encloses no singular point other than z_o.*

- *Let 'a' is a singular point of f(z), then*
  (a) *if $\lim_{z\to a} f(z)$ exists and is finite, then 'a' is removable singularity.*
  (b) *if $\lim_{z\to a} f(z)$ exists infinitely, then 'a' is a pole.*
  (c) *if $\lim_{z\to a} f(z)$ does not exist, then 'a' is essential singularity.*

- *The principle part of Laurent series expansion of f(z) about point 'a', where a is a singular point of f(z), has no terms if 'a' is removable singularity, finitely many terms if 'a' is pole and infinitely many terms if 'a' is essential singularity.*

- *Residue of f(z) at z=a is the coefficient of $\frac{1}{z}$ in the Laurent series expansion of f(z) about a singularity 'a'*

- *Residue of a function f(z) at a pole z=a is calculated as follows*
  $\operatorname*{Res}_{z=a} f(z)=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$, *where n is the order of the pole*

- ***Residue Theorem:** let f(z) be single valued and analytic inside and on a simple closed curve C except at all singularities a,b,c,... inside C, then residue theorem states that*
  $\oint_C f(z)dz=2\pi i\ [\operatorname*{Res}_{z=a} f(z)+\operatorname*{Res}_{z=b} f(z)+\operatorname*{Res}_{z=c} f(z)+...]$

- ***Argument Theorem:** let f(z) be analytic inside and on a simple closed curve except for a finite number poles inside C. Then $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz=N-P$, where N is the number of zeros of f(z) inside C and P is the number of poles inside C(including multiplicities)*

- ***Identity Theorem:** Suppose that f is analytic to a domain D. If the set of zeros of f has a limit in D, then $f\equiv 0$ in D.*

- ***Open Mapping Theorem:** A non-constant analytic function on a connected domain maps open sets to open sets.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. The value of $\int_0^{2\pi} \exp(e^{i\theta} - i\theta)\, d\theta$ equals (GATE-2006)

(a) $2\pi i$ (b) 2π (c) π (d) $i\pi$

Solution: (b) Let $I = \int_0^{2\pi} \exp(e^{i\theta} - i\theta)\, d\theta = \int_0^{2\pi} e^{e^{i\theta}} e^{-i\theta} d\theta$

Let C be the circle $|z| = 1 \Rightarrow z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$\therefore I = \int_C e^z \frac{dz}{iz^2} = \frac{1}{i}\int_C \frac{e^z}{z^2} dz = \frac{1}{i}\int_C f(z)dz$ ...(1), where $f(z) = \frac{e^z}{z^2}$

Poles of $f(z)$ are given by $z^2 = 0 \Rightarrow z = 0,0$

$\Rightarrow z = 0$ is a pole of order 2 $\quad \therefore \text{Res}(z=0) = \lim_{z\to 0}\left[\frac{d}{dz}(z-0)^2 \frac{e^z}{z^2}\right] = 1$

By Cauchy's Integral Formula, we have $\int_C f(z)\, dz = 2\pi i$ (sum of residues within C) $= 2i\pi(1) = 2i\pi$

$\therefore$ (1) becomes, $I = \frac{1}{i} \times 2i\pi = 2\pi$

So, option (b) is correct.

Example 2. The sum of the residues at all the poles of $f(z) = \frac{\cot \pi z}{(z+a)^2}$, where a is constant, ($a \neq 0, \pm 1, \pm 2, \ldots$) is (GATE-2006)

(a) $\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi \operatorname{cosec}^2 \pi a$

(b) $-\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi \operatorname{cosec}^2 \pi a$

(c) $-\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi \operatorname{cosec}^2 \pi a$

(d) $\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi \operatorname{cosec}^2 \pi a$

Solution: (a) Poles of $f(z)$ are given by $\sin \pi z = 0$ and $(z+a)^2 = 0$, which gives $z = -a$ is pole of order two and $z = n$ is pole of order one, where $n = 0, \pm 1, \pm 2, \ldots$

Residue at ($z=-a$) $= \lim_{z\to -a} \frac{d}{dz}\left((z+a)^2 \frac{\cot \pi z}{(z+a)^2}\right) = -\pi \operatorname{cosec}^2 \pi a$

Residue at ($z = n$) $= \lim_{z\to n} \frac{(z-n)\cos \pi z}{(z+a)^2 \sin \pi z} = \frac{1}{\pi}\frac{1}{(n+a)^2}$

$\therefore$ *Sum of residues at all poles of* $f(z)$ *is given by* $\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}-\pi\operatorname{cosec}^2\pi a$

Example 3. *Let* $S=\{0\}\cup\left\{\frac{1}{4n+7}:n=1,2,\ldots\right\}$. *Then the number of analytic functions which vanish only on S is* **(GATE-2007)**

(a) infinite *(b) 0* *(c) 1* *(d) 2*

Solution: (b) $S=\{0\}\cup\left\{\frac{1}{4n+7}:n\in\mathbb{N}\right\}$, *if f(z) is analytic function and its zeros are 0,* $\frac{1}{4n+7}, n\in\mathbb{Z}$

We know that limit point of zeros is an isolated essential singularity.

Here, 0 is limit point of $\frac{1}{4n+7}$; $n\in\mathbb{Z}$

$\Rightarrow$ *0 is an isolated essential singularity* $\Rightarrow$ *0 cannot be zero (root) of f(z)*

$\Rightarrow$ *There is no such analytic function which vanish only on S*

$\Rightarrow$ *option (b) is correct.*

Example 4. *Let S be the disk |z|<3 in the complex plane and let* $f:S\to\mathbb{C}$ *be an analytic function such that* $f\left(1+\frac{\sqrt{2}}{n}i\right)=-2/n^2$ *for each natural number n. Then* $f(\sqrt{2})$ *is equal to* **(GATE-2008)**

(a) $3-2\sqrt{2}$ *(b)* $3+2\sqrt{2}$ *(c)* $2-3\sqrt{2}$ *(d)* $2+3\sqrt{2}$

Solution: (a) *Defining* $f(z)=(z-1)^2$

Then $f\left(1+\frac{\sqrt{2}}{n}i\right)=-\frac{2}{n^2}\ \forall\ n\in\mathbb{N}$

Also, f is an analytic function.

$\therefore f(\sqrt{2})=(\sqrt{2}-1)^2=3-2\sqrt{2}$

Example 5. *Let* $f(z)=\cos z-\frac{\sin z}{z}$ *for non-zero* $z\in\mathbb{C}$ *and* $f(0)=0$. *Then f(z) has a zero at z = 0 of order* **(GATE-2008)**

(a) 0 *(b) 1* *(c) 2* *(d) greater than 2*

Solution: (c) *As* $f(z)=\cos z-\frac{\sin z}{z}=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right)-\frac{\sin z}{z}$

$$=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right)-\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right)=z^2\left[\left(\frac{1}{3!}-\frac{1}{2!}\right)-z^2\left(\frac{1}{5!}-\frac{1}{4!}\right)+z^4\left(\frac{1}{7!}-\frac{1}{6!}\right)-\ldots\right]$$

$\therefore$ *f(z) has a zero of order 2 at z=0*

Example 6. Let $f(z) = \cos z - \dfrac{\sin z}{z}$ for non-zero $z \in \mathbb{C}$. Also let $g(z) = \sinh z$ for $z \in \mathbb{C}$

Then $\dfrac{g(z)}{zf(z)}$ has a pole at $z = 0$ of order (GATE-2008)

(a) 1 (b) 2 (c) 3 (d) greater than 3

Solution: (b) $\dfrac{g(z)}{zf(z)} = \dfrac{\sinh z}{z\cos z - \sin z} = \dfrac{e^z - e^{-z}}{2(z\cos z - \sin z)}$

$$= \frac{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots}{z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)}$$

$$= \frac{z\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots\right)}{z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots - 1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \ldots\right)}$$

$$= \frac{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots}{z^2\left\{\left(\frac{1}{3!} - \frac{1}{2!}\right) - z^2\left(\frac{1}{5!} - \frac{1}{4!}\right) + z^4\left(\frac{1}{7!} - \frac{1}{6!}\right) - \ldots\right\}}$$

$\therefore \dfrac{g(z)}{zf(z)}$ has pole at $z=0$ of order 2

Example 7. Let S be the positively oriented circle given by $|z - 3i| = 2$. Then the value of $\int_S \dfrac{dz}{z^2 + 4}$ is (GATE-2008)

(a) $\dfrac{-\pi}{2}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{-i\pi}{2}$ (d) $\dfrac{i\pi}{2}$

Solution:(b) $I = \int_S \dfrac{dz}{z^2 + 4} = \int_S f(z)dz$, where $f(z) = \dfrac{1}{z^2 + 4}$

Poles of $f(z)$ are given by

$z^2 + 4 = 0 \Rightarrow z = \pm 2i$

$z = 2i$ lies inside S

$$\therefore \text{Res}(z = 2i) = \lim_{z \to 2i}(z - 2i)\frac{1}{(z - 2i)(z + 2i)} = \frac{1}{4i}$$

By Cauchy Residue Theorem, $\int_S f(z)dz = 2\pi i \times$ sum of residue within $S = 2\pi i\left(\dfrac{1}{4i}\right) = \dfrac{\pi}{2}$

Example 8. *For the function* $f(z) = \sin\left(\frac{1}{\cos(1/z)}\right)$, *the point* $z = 0$ *is* *(GATE-2009)*

(a) a removable singularity *(b) pole*
(c) an essential singularity *(d) a non-isolated singularity.*

Solution: (c) $f(z) = \sin\left(\frac{1}{\cos(1/z)}\right)$

Now, $\sin\left(\frac{1}{\cos(1/z)}\right) = 0 \Rightarrow \cos\left(\frac{1}{z}\right) = \frac{1}{n\pi},\ n \in \mathbb{Z}$

Also, $\cos(1/z) = 0$

$\Rightarrow \frac{1}{z} = \left(\frac{\pi}{2} + n\pi\right),\ n \in \mathbb{Z} \Rightarrow z = \frac{2}{(2n+1)\pi},\ n \in \mathbb{Z}$

So, as $n \to \infty$, $z = 0$.

Hence, $z = 0$ *is an essential singularity.*

Example 9. *Consider the function* $f(z) = \frac{e^{iz}}{z(z^2+1)}$. *The residue of f at the isolated singular points in the upper half plane* $\{z = x + iy \in \mathbb{C} : y > 0\}$, *is* *(GATE-2009)*

(a) $\frac{-1}{2e}$ *(b)* $\frac{-1}{e}$ *(c)* $\frac{e}{2}$ *(d) 1*

Solution: (a) *As* $f(z)$ *has isolated singularities at* $z = 0, i, -i$ *of which* $z = i$ *lies in the upper half plane.*

$\therefore$ *Residue of f at* $z = i = \lim_{z\to i}(z-i)\frac{e^{iz}}{z(z^2+1)} = \lim_{z\to i}\frac{e^{iz}}{z(z+i)} = \frac{e^{-1}}{-2} = -\frac{1}{2e}$

Example 10. *Let C be the contour* $|z|=2$ *oriented in the anti-clockwise direction. The value of the integral* $\oint_C ze^{3/z}dz$ *is* *(GATE-2013)*

(a) $3\pi i$ *(b)* $5\pi i$ *(c)* $7\pi i$ *(d)* $9\pi i$

Solution: (d) $\oint_C ze^{\frac{3}{z}}dz$, *where C is* $|z|=2$

$$ze^{3/z} = z\left(1 + \frac{3}{z} + \frac{9}{z^2 2!} + \ldots\right) = z + 3 + \frac{9}{2}\cdot\frac{1}{z} + \ldots$$

Residue at (z=0) $= \frac{9}{2}$

$$\oint_{|z|=2} ze^{\frac{3}{z}}dz = 2\pi i \times \frac{9}{2} = 9\pi i$$

Example 11. *The value of* $\frac{i}{4-\pi}\int_{|z|=4}\frac{dz}{z\cos(z)}$ *is equal to*______. *(GATE-2015)*

Solution: *Singularities of* $f(z)=\frac{1}{z\cos z}$ *inside* $|z| = 4$ *are at* $z=0, \frac{\pi}{2}, -\frac{\pi}{2}$

Residue at $z=0$, $\lim_{z\to 0}\frac{z}{z\cos z}=1$

Residue at $z=\frac{\pi}{2}$, $\lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)}{z\cos z}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{z\to\frac{\pi}{2}}\frac{1}{\cos z - z\sin z}=-\frac{2}{\pi}$

Residue at $z=-\frac{\pi}{2}$, $\lim_{z\to-\frac{\pi}{2}}\frac{\left(z+\frac{\pi}{2}\right)}{z\cos z}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{z\to-\frac{\pi}{2}}\frac{1}{\cos z - z\sin z}=\frac{1}{-\left(-\frac{\pi}{2}\right)(-1)}=-\frac{2}{\pi}$

$$\int_{|z|=4}\frac{dz}{z\cos z}=2\pi i\left(1-\frac{2}{\pi}-\frac{2}{\pi}\right)=2\pi i\left(\frac{\pi-4}{\pi}\right)$$

$$\frac{i}{4-\pi}\int_{|z|=4}\frac{dz}{z\cos z}=\left(\frac{i}{4-\pi}\right)2\pi i\left(\frac{\pi-4}{\pi}\right)=2$$

Example 12. *Let (z_n) be a sequence of distinct points in $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ with* $\lim_{n\to\infty} z_n = 0$.

Consider the following statements P and Q :

P : *There exists a unique analytic function f on $D(0,1)$ such that $f(z_n) = \sin(z_n)$ for all n.*

Q : *There exists an analytic function f on $D(0,1)$ such that $f(z_n) = 0$ if n is even and $f(z_n) = 1$ if n is odd.*

Which of the above statements hold TRUE? *(GATE-2016)*

(a) Both P and Q *(b) Only P* *(c) Only Q* *(d) Neither P nor Q*

Solution: (b) *(P) is true*

Take $g(z) = \sin z$

$(f-g)(z_n) = 0$ *for all n*

By using Identity theorem, we get $f-g\equiv 0 \Rightarrow f(z)\equiv \sin z. \Rightarrow f$ *is unique.*

(Q) is false, as f is analytic $\Rightarrow f$ *is continuous. If* $z_n \to 0 \Rightarrow f(z_n) \to f(0)$, *all its subsequences also go to* $f(0) \Rightarrow 0 = f(0) = 1$, *contradiction*

Example 13. *Let $f : D \to D$ be holomorphic with $f(0)=1/2$ and $f(1/2)=0$, where $D = \{z : |z| \leq 1\}$. Which of the following is correct?* ***(CSIR UGC NET JUNE-2011)***

(a) $|f'(0)| \leq 3/4$

(b) $|f'(1/2)| \leq 4/3$

(c) $|f'(0)| \leq 3/4$ *and* $|f'(1/2)| \leq 4/3$

(d) $f(z) = z, z \in D$

Solution: (a, b, c) $f : D \to D$ *is holomorphic with* $f(0) = \frac{1}{2}$ *and* $f\left(\frac{1}{2}\right) = 0$, *where $D = \{z : |z| \leq 1\}$*

According to Schwarz pick theorem, $|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$

$$\text{Now } |f'(0)| \leq \frac{1-|f(0)|^2}{1-|0|^2} = \frac{1-\left|\left(\frac{1}{2}\right)\right|^2}{1} \quad \left\{\because f(0) = \frac{1}{2}\right\}$$

$$\Rightarrow |f'(0)| \leq 1 - \frac{1}{4} = \frac{3}{4} \qquad \Rightarrow |f'(0)| \leq \frac{3}{4}$$

$\Rightarrow$ *option (a) is correct*

$$\text{Now} \left|f'\left(\frac{1}{2}\right)\right| \leq \frac{1-\left|f\left(\frac{1}{2}\right)\right|^2}{1-\left|\frac{1}{2}\right|^2} = \frac{1-0}{1-\frac{1}{4}} \quad \Rightarrow \left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$$

$\Rightarrow$ *Option (a), (b) and (c) are correct*

Clearly option (d) is incorrect

$\because$ *if* $f(z) = z$

$\Rightarrow f(0) = 0$ *and* $f\left(\frac{1}{2}\right) = \frac{1}{2}$, *which is contradiction to given condition.*

Example 14. *At z=0 the function* $f(z) = \frac{e^z+1}{e^z-1}$ *has* ***(CSIR UGC NET JUNE-2011)***

(a) *a removable singularity*

(b) *a pole*

(c) *an essential singularity*

(d) *the residue of f(z) at z=0 is 2.*

Solution: (b,d) $f(z) = \frac{e^z+1}{e^z-1} = \frac{p(z)}{q(z)}$

At z=0, $f(z)$ *has a pole* $\left\{\because \lim_{z\to 0} f(z) = \frac{2}{0} = \infty\right\}$.

So, option (b) is correct.

As $e^0 = 1$, $e^z - 1 = 0$

$\therefore$ *Res (z=0)* $= \dfrac{p(0)}{q'(0)}$

$=\dfrac{e^0+1}{e^0}=2$

$\therefore$ *Residue of $f(z)$ at $z=0$ is 2*

$\therefore$ *option (d) is correct*

Example 15. *Let f be an entire function. If Ref is bounded, then* **(CSIR UGC NET JUNE-2011)**

(a) $\text{Im } f$ *is constant* (b) f *is constant*

(c) $f\equiv 0$ (d) f' *is a non zero constant*

Solution: (a,b) *Given f is an entire function such that Ref is bounded $\Rightarrow$ 'f' omits uncountably many values.*

Picard's theorem:- *Every non-constant entire function omits at most one complex number as its value or we can say, if an entire function omits two or more values, then it is constant.*

$\Rightarrow$ *f is constant* $\Rightarrow$ *Imf is constant*

Thus options (a) and (b) are correct.

Example 16. *Let f be entire function such that $\lim_{|z|\to\infty}|f(z)|=\infty$. Then* **(CSIR UGC NET DEC-2011)**

(a) $f\left(\dfrac{1}{z}\right)$ *has an essential singularity at 0* (b) f *cannot be a polynomial*

(c) f *has finitely many zeros* (d) $f\left(\dfrac{1}{z}\right)$ *has a pole at 0*

Solution: (c,d) *Given that f is entire function and* $\lim_{|z|\to\infty}|f(z)|=\infty$

Let $f(z)=z$

Clearly, f is entire and $\lim_{|z|\to\infty}|f(z)|=\infty$

$\Rightarrow$ *Option (b) is incorrect, as $f(z)=z$ is polynomial and* $f\left(\dfrac{1}{z}\right)=\dfrac{1}{z}$, *has pole at $z=0$*

$\Rightarrow$ *option (a) is also incorrect*

Similarly we can choose any polynomial of finite degree i.e. $f(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n$

$\lim_{|z|\to\infty}|f(z)|=\infty$ *and* $f\left(\dfrac{1}{z}\right)=a_0+\dfrac{a_1}{z}+\dfrac{a_2}{z^2}+\ldots+\dfrac{a_n}{z^n}$

Clearly $f\left(\dfrac{1}{z}\right)$ *has singularity at $z=0$ and principle part of $f(z)$ has finite number of terms*

$\Rightarrow f\left(\dfrac{1}{z}\right)$ *has a pole at $z=0$* $\Rightarrow$ *options (c) and (d) are correct.*

Example 17. *Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function and let $g:\mathbb{C}\to\mathbb{C}$ be defined by $g(z)=f(z)-f(z+1)$ for $z\in\mathbb{C}$. Which of the following statements are true?* **(CSIR UGC NET DEC-2011)**

(a) If $f\left(\frac{1}{n}\right)=0$ for all positive integers n, then f is a constant function.

(b) If $f(n)=0$ for all positive integers n, then f is a constant function.

(c) If $f\left(\frac{1}{n}\right)=f\left(\frac{1}{n}+1\right)$ for all positive integers n, then g is a constant function.

(d) If $f(n)=f(n+1)$ for all positive integers n, then g is a constant function.

Solution: (a,c) *We have $f:\mathbb{C}\to\mathbb{C}$ is an entire function*

For option (a),

$f\left(\frac{1}{n}\right)=0$ i.e., $f(1)=0$

$f\left(\frac{1}{2}\right)=0$

$f\left(\frac{1}{3}\right)=0$

$\vdots$

Also f is analytic at '0'

$\Rightarrow f$ is analytic at limit point of zeros of f(z)

By Identity theorem, $f\equiv 0$ in $\mathbb{C}$

So option (a) is correct.

For Option (b), $f(n)=0\ \forall\, n\in\mathbb{Z}^+$

limit point of 1, 2, 3, 4, ..., i.e., zeros of f(z) is ∞ and f is not analytic at $z=\infty$

So we can't say f is constant function

For option (c), $f\left(\frac{1}{n}\right)=f\left(1+\frac{1}{n}\right)\forall\, n\in\mathbb{Z}^+$ we have $g(z)=f(z)-f(1+z)\ \forall\, z\in\mathbb{C}$

$\Rightarrow g\left(\frac{1}{n}\right)=f\left(\frac{1}{n}\right)-f\left(1+\frac{1}{n}\right)$

$g(1)=0$

$g\left(\frac{1}{2}\right)=0$

$g\left(\frac{1}{3}\right)=0$

$\vdots$

Also g is analytic at '0' and '0' is limit point of zeros of g(z)

$\Rightarrow g\equiv 0$ in $\mathbb{C}$

$\Rightarrow$ *option (c) is correct*

For option (d), $f(n) = f(n+1)\ \forall\ n \in \mathbb{Z}^+$

We have $g(z) = f(z) - f(1+z)\ \forall\ z \in \mathbb{C}$

$\Rightarrow g(1) = 0$

$g(2) = 0$

$g(3) = 0$

$\vdots$

limit point of 1, 2, 3, ..., i.e., zeros of g(z) is ∞

But g is not analytic at $z = \infty$

So we can't say g is constant function

$\Rightarrow$ *only options (a) and (c) are correct.*

Example 18. *At* $z = 0$, *the function* $f(z) = \exp\left(\frac{z}{1-\cos z}\right)$ *has* **(CSIR UGC NET JUNE-2012)**

(a) a removable singularly

(b) a pole

(c) an essential singularity

(d) the Laurent expansion of $f(z)$ *around* $z=0$ *has infinitely many positive and negative powers of* z.

Solution: (c, d) $\lim_{z\to 0} f(z) = \lim_{z\to 0} \exp\left(\frac{z}{1-\cos z}\right) = \exp\left(\lim_{z\to 0}\frac{z}{1-\cos z}\right)$

$= \exp\left(\lim_{z\to 0}\frac{1}{-\sin z}\right) = e^{\infty}$ *=does not exists*

Hence, $f(z) = \exp\left(\frac{z}{1-\cos z}\right)$ *has an essential singularity.*

$$\exp\left(\frac{z}{1-\cos z}\right) = 1 + \left(\frac{z}{1-\cos z}\right) + \left(\frac{z}{1-\cos z}\right)^2\frac{1}{2!} + \left(\frac{z}{1-\cos z}\right)^3\frac{1}{3!} + \dots$$

$$= 1 + \frac{z}{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)} + \left(\frac{z}{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)}\right)^2\frac{1}{2!} + \dots$$

Hence, options (c) and (d) are correct.

Example 19. *Consider the functions f, g :* $\mathbb{C} \to \mathbb{C}$ *defined by* $f(z) = e^z$ *and* $g(z)=e^{iz}$.

Let $S=\{z \in \mathbb{C} : \mathrm{Re}\, z \in [-\pi,\pi]\}$. *Then* **(CSIR UGC NET DEC-2012)**

(a) f is an onto entire function.

(b) g is a bounded function on $\mathbb{C}$.

(c) f is bounded on S.

(d) g is bounded on S.

Solution: (c) $f, g : \mathbb{C} \to \mathbb{C}$ and $f(z) = e^z, g(z) = e^{iz}$ and $S = \{z \in \mathbb{C} : \text{Re } z \in [-\pi, \pi]\}$

$f(z) = e^{x+iy}$

$|f(z)| = |e^{x+iy}| = |e^x||e^{iy}| = e^x \; \{\because |e^{iy}| = 1\}$

$|f(z)| = e^x$ as $-\pi \le x \le \pi$

$\Rightarrow f(z)$ is bounded on S.

$\therefore$ option (c) is correct.

$g(z) = e^{i(x+iy)} = e^{-y} \cdot e^{ix}$

$\therefore |g(z)| = |e^{-y} \cdot e^{ix}| = e^{-y}$

Clearly, $g(z)$ is not bounded on S and hence not on $\mathbb{C}$

Example 20. *Which of the following functions f are entire functions and have simple zeros at $z = ik$ for all $k \in \mathbb{Z}$.*

(CSIR UGC NET DEC-2012)

(a) $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ for some $n \ge 1$ and some $a_0, a_1, \ldots, a_n \in \mathbb{C}$.

(b) $f(z) = a \sin 2\pi i z$, for some $a \in \mathbb{C}$.

(c) $f(z) = b \cos 2\pi (iz - 1/4)$, for some $b \in \mathbb{C}$.

(d) $f(z) = e^{cz}$, for some $c \in \mathbb{C}$.

Solution: (b,c) We have to find these functions from options which are entire and have simple zeros at $z = ik \; \forall \; k \in \mathbb{Z}$

For option (a), $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $n \ge 1$ and $a_0, a_1, \ldots, a_n \in \mathbb{C}$

$f(ik) = a_n (ik)^n + a_{n-1} (ik)^{n-1} + \ldots + a_0$

$\Rightarrow f(ik) \neq 0$

$\therefore$ Option (a) is incorrect

For option (b),

$f(z) = a \sin 2\pi iz$

$f(ik) = a \sin 2\pi i (ik) = a \sin(-2\pi k)$

$\Rightarrow f(ik) = -a \sin 2\pi k = 0 \; \forall \; k \in \mathbb{Z} \Rightarrow z = ik$ is zero if $f(z) = a \sin 2\pi iz$

now $f'(z) = a \cos 2\pi iz \; 2\pi i$

$\Rightarrow f'(ik) = 2\pi i \cdot a \cos 2\pi i (ik) = 2\pi ai \cos 2\pi k$

$\Rightarrow f'(ik) = 2\pi ai \cos 2\pi k \neq 0 \quad \forall \; k \in \mathbb{Z}$

Also $f(z)$ is analytic

Thus, option (b) is correct

For option (c),

$f(z) = b \cos 2\pi \left(iz - \frac{1}{4} \right), b \in \mathbb{C}$

$$f(ik) = b\cos 2\pi\left(i.ik - \frac{1}{4}\right) = b\cos 2\pi\left(-k - \frac{1}{4}\right)$$

$$f(ik) = b\cos 2\pi\left(k + \frac{1}{4}\right) = b\cos(4k+1)\frac{\pi}{2} = 0 \text{ for } k \in \mathbb{Z} \Rightarrow ik \text{ is zero of } f(z) = b\cos 2\pi\left(iz - \frac{1}{4}\right)$$

$$f'(ik) = -b.2\pi i\sin 2\pi\left(-k - \frac{1}{4}\right)$$

$$f'(ik) = b2\pi i\sin(4k+1).\frac{\pi}{2} \neq 0,\ k \in \mathbb{Z}$$

Also $f(z) = b\cos 2\pi\left(iz - \frac{1}{4}\right)$ *is analytic*

Thus, option (c) is correct

For option (d), $f(z) = e^z$

$$f(ik) = e^{ik} = \cos k + i\sin k \neq 0, k = 1 \in \mathbb{Z}$$

$\Rightarrow$ *option (d) is incorrect*

Example 21. *Let f be an analytic function defined on* $D = \{z \in \mathbb{C} : |z| < 1\}$ *such that the range of f is contained in the set* $\mathbb{C} \setminus (-\infty, 0]$*. Then.* **(CSIR UGC NET DEC-2012)**

(a) *f is necessarily a constant function.*

(b) *there exists an analytic function g on D such that g(z) is a square root of f (z) for each* $z \in D$.

(c) *there exists an analytic function g on D such that Re* $g(z) \geq 0$ *and g(z)is a square root of f (z) for each* $z \in D$.

(d) *there exists an analytic function g on D such that Re* $g(z) \leq 0$ *and g(z)is a square root of f (z) for each* $z \in D$.

Solution: (b, c, d) *Given ,* $f : D \longrightarrow \mathbb{C}\setminus(-\infty, 0]$ *is analytic, where* $D = \{z \in \mathbb{C} : |z| < 1\}$

Take, $f(z) = z^4$

Here, $g(z) = z^2$

Clearly, g(z) is analytic

Hence option (b) is correct and therefore option (a) is incorrect

Further, take $f(z) = 4$

$$g(z) = \sqrt{f(z)} = \pm 2$$

Take g(z) = 2, then option (c) is correct

Take g(z) = -2, then option (d) is correct

Example 22. *Let f be a non-constant entire function. Which of the following properties is possible for f for each* $z \in \mathbb{C}$? **(CSIR UGC NET DEC-2013)**

(a) $\text{Re}\, f(z) = \text{Im}\, f(z)$ (b) $|f(z)| < 1$ (c) $\text{Im}\, f(z) < 0$ (d) $f(z) \neq 0$

Solution: (d) Result: *Picard's theorem states that a non-constant entire function cannot omit more than one value or we can say if an entire function omits more than one value then it is constant.*
Using above result options (a), (b), (c) are incorrect.
As all other options are incorrect
$\therefore$ *option (d) is correct*

Example 23. *Let f be a holomorphic function on the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Which of the following is/are necessarily true?* ***(CSIR UGC NET DEC-2013)***

(a) *If for each positive integer n we have* $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$, *then $f(z) = z^2$ on the unit disc.*

(b) *If for each positive integer n we have* $f\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{n}\right)^2$, *then $f(z) = z^2$ on the unit disc*

(c) *f cannot satisfy* $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ *for each positive integer n.*

(d) *f cannot satisfy* $f\left(\frac{1}{n}\right) = \frac{1}{n+1}$ *for each positive integer n.*

Solution: (a, c)

If f is a homomorphic function and satisfies $f\left(\frac{1}{n}\right) = \frac{1}{n^2} \quad \forall n \in \mathbb{Z}^+$

For option (a),

Let $g(z) = z^2$

Consider a new function

$h(z) = f(z) - g(z)$

Clearly $h(z)$ *has zeros at* $z = \frac{1}{n}, n \in \mathbb{Z}^+$

Also $h(z)$ *is holomorphic function as we have given that f is holomorphic function.*

$\Rightarrow h(z)$ *is also analytic at limit point of zeros of h(z), i.e., at z = 0*

$\Rightarrow h(z) \equiv 0$ *(By Identity Theorem)*

$\Rightarrow f(z) = g(z) = z^2$ *on the unit disc*

So option (a) is correct

For option (b),

$f(z)$ *is a holomorphic function and satisfies*

$$f\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{n}\right)^2$$

Consider $g(z) = z^2$.

Consider a new function

$h(z) = f(z) - g(z)$

As $h(z)$ is analytic and has zeroes at $\left(1-\frac{1}{n}\right)$, $n\in\mathbb{Z}^+$ but $h(z)$ is not given analytic at the limit point of its zeros, i.e., at z = 1. So identity theorem is not applicabale.
For option (c),

f *is a holomorphic function on the unit disc and satisfies* $f\left(\frac{1}{n}\right)=\frac{(-1)^n}{n}\ \forall n\in\mathbb{Z}^+$

Let $g(z)=z$

Consider $h(z)=f(z)-g(z)=f(z)-z$

As $f(z)$ is analytic on the unit disc and $h(z)$ has zeros at $z=\frac{1}{2n}$ and $h(z)$ is analytic at the limit point of its zeros ,i.e.,at $z=0$.

$\Rightarrow h(z)=0$ *on the unit disc (by identity theorem)*

$\Rightarrow f(z)=z$ *on the unit disc* ...(1)

Also, let $h_1(z)=f(z)+g(z)=f(z)+z$

As h(z) has zeros at $z=\frac{-1}{2n}$ *and h(z) is analytic at z = 0*

$\Rightarrow h_1(z)=0$ *on unit disc*

which is not possible $\Rightarrow f(z)=-z$ ...(2)

For option (d), take $f(z)=\frac{z}{1+z}$

Clearly, f is analytic in unit disc and $f\left(\frac{1}{n}\right)=\frac{1}{n+1}$

$\Rightarrow$ *There exist* f *such that* f *satisfies* $f\left(\frac{1}{n}\right)=\frac{1}{n+1}$ *for each positive integer* n.

$\Rightarrow$ *option (d) is incorrect.*

Example 24. *Let f,g be meromorphic functions on $\mathbb{C}$. If f has a zero of order k at z=a and g has a pole of order m at z=0, then g(f(z)) has* ***(CSIR UGC NET JUNE-2014)***

(a) a zero of order km at z=a
(b) a pole of order km at z=a
(c) a zero of order |k-m| at z=a
(d) a pole of order |k-m| at z=a

Solution: (b) *Here, f has a zero of order k at z=a and g has a pole of order m at z=0.*

Let $f(z)=(z-a)^k$ *and* $g(z)=\frac{1}{(z-0)^m}=\frac{1}{z^m}$

Then $g(f(z))=g[(z-a)^k]=\frac{1}{[(z-a)^k]^m}=\frac{1}{(z-a)^{km}}$

So we have $g(f(z))$ *has a pole of order km at z=a.*

$\therefore$ *Option (b) is correct.*

Example 25. *For $z \in \mathbb{C}$, define $f(z)=\dfrac{e^z}{e^z-1}$. Then* **(CSIR UGC NET JUNE-2014)**

(a) f is entire.
(b) the only singularities of f are poles.
(c) f has infinitely many poles on the imaginary axis.
(d) each pole of f is simple.

Solution: (b,c,d) We have $f(z)=\dfrac{e^z}{e^z-1}$

For singularities of $f(z)$, take $e^z-1=0 \Rightarrow e^z=1=e^{2n\pi i} \Rightarrow z=2n\pi i,\ n\in\mathbb{Z}$

Since, $\lim\limits_{z\to 2n\pi i} f(z)=\infty$ and $\lim\limits_{z\to 2n\pi i}(z-2n\pi i)f(z)=1$

So $f(z)$ has simple poles at $z=2n\pi i,\ n\in\mathbb{Z}$

$\Rightarrow$ f has infinitely many poles on the imaginary axis and each pole of f is simple

$\Rightarrow$ Options (b), (c), (d) are correct

$f(z)$ has infinitely many singularities

$\Rightarrow$ f is not entire

$\Rightarrow$ Option (a) is incorrect.

Example 26. *Let $p(z)=a_0+a_1z+\ldots+a_nz^n$ and $q(z)=b_1z+b_2z^2+\ldots+b_nz^n$ be complex polynomials. If a_0, b_1 are non-zero complex numbers, then the residue of $p(z)/q(z)$ at 0 is equal to*

(CSIR UGC NET DEC-2014)

(a) $\dfrac{a_0}{b_1}$ (b) $\dfrac{b_1}{a_0}$ (c) $\dfrac{a_1}{b_1}$ (d) $\dfrac{a_0}{a_1}$

Solution: (a)

We know that, if $f(z)=\dfrac{\phi(z)}{\psi(z)},\phi(a)\neq 0$, and at the point $z=a,\psi(z)$ has a zero of order one, i.e.,

$\psi(a)=0,\psi'(a)\neq 0$, then $\operatorname*{res}\limits_{z=a} f(z)=\dfrac{\phi(a)}{\psi'(a)}$.

Here given that $p(z)=a_0+a_1z+\ldots+a_nz^n$ and $q(z)=b_1z+b_2z^2+\ldots+b_nz^n$

Therefore the residue of $\dfrac{p(z)}{q(z)}$ at 0 is equal to $\operatorname*{res}\limits_{z=0}\left(\dfrac{p(z)}{q(z)}\right)=\dfrac{p(0)}{q'(0)}=\dfrac{a_0}{b_1}$

Hence, option (a) is correct.

Example 27. *Let f be an entire function. Which of the following statements are correct?*

(CSIR UGC NET JUNE-2015)

(a) f is constant if the range of f is contained in a straight line.
(b) f is constant if f has uncountably many zeros.

(c) f is constant iff f is bounded on $\{z \in \mathbb{C}: Re(z) \leq 0\}$.
(d) f is constant if the real part of f is bounded.

Solution: (a,b,d)

According to Picard's theorem, every non-constant entire function omit at most one value in complex plane.

For option (a),
Range of f is contained in a straight line. Range of f is bounded. So by Picard's theorem, f is constant.

$\therefore$ *option (a) is correct*

For option (b),
Non-constant entire function has always countably many zeros.
Only entire function that has uncountably many zeros is $f(z) = 0$.
Thus, option (b) is correct.

For option (c),
If f is bounded on $\{z \in \mathbb{C} : Re(z) \leq 0\}$, then we can't apply Picard's theorem

$\because$ *for Picard's theorem function should be bounded for whole complex plane.*

Take $f(z) = e^{-z} = e^{-x} . e^{-iy}$

$\therefore$ $|f(z)| = |e^{-x} . e^{-iy}| = e^{-x}$

Clearly, $f(z)$ is bounded on $\{z \in \mathbb{C} : Re\, z \leq 0\}$, but $f(z)$ is non-constant
Thus, option (c) is incorrect.

For option (d), If the real part of f is bounded, then f is bounded for whole complex plane. So, by Picard's theorem, f is a constant function.
Thus, option (d) is correct.

Example 28. *Let f be an analytic function in $\mathbb{C}$. Then f is constant if the zero set of f contains the sequence* ***(CSIR UGC NET DEC-2015)***

(a) $a_n = 1/n$

(b) $a_n = (-1)^{n-1}\frac{1}{n}$

(c) $a_n = \frac{1}{2n}$

(d) $a_n = n$ if 4 does not divide n and $a_n = \frac{1}{n}$ if 4 divides n

Solution: (a,b,c,d) *Given that f is analytic in $\mathbb{C}$*

For option (a), If $a_n = \frac{1}{n}$

limit point of zeros is '0' which is in the domain where f is analytic.

$\therefore$ *By identity theorem, $f \equiv 0$*

For option (b),

$a_n = (-1)^{n-1}\frac{1}{n}$

limit point of zeros is '0' $\Rightarrow$ $f \equiv 0$ *(By identity theorem)*

For option (c),

$$a_n = \frac{1}{2n} \to 0 \Rightarrow f \equiv 0 \text{ (By identity theorem)}$$

For option (d),

$$a_n = \begin{cases} n, \text{ if } 4 \nmid n \\ \frac{1}{n}, \text{ if } 4 \mid n \end{cases}$$

$\Rightarrow$ *We get a subsequence* $\frac{1}{n}$ *for which* $f\left(\frac{1}{n}\right) = 0$ *and limit point of zeros is 0.*

$\therefore$ *using same argument, we get* $f \equiv 0$

Thus, all options are correct.

Example 29. *The residue of the function* $f(z) = e^{-e^{1/z}}$ *at z=0 is* **(CSIR UGC NET JUNE-2016)**

(a) $1+e^{-1}$ (b) e^{-1} (c) $-e^{-1}$ (d) $1-e^{-1}$

Solution. (c) Given $f(z) = e^{-e^{\frac{1}{z}}}$

To find Residue at z = 0,

We have to find coefficient of $\frac{1}{z}$.

$$\text{Now } e^{-e^{1/z}} = e^{-\left[1+\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\frac{1}{4!}\left(\frac{1}{z}\right)^4+\ldots\right]}$$

$$=1-\left[1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\ldots\right]+\frac{1}{2!}\left[1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\ldots\right]^2-\frac{1}{3!}\left[1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\ldots\right]^3+\ldots$$

$$\left[\because e^{-z} = 1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\ldots\right]$$

$$=\frac{1}{z}\left(-1+\frac{2}{2!}-\frac{3}{3!}+\frac{4}{4!}-\ldots\right)+\ldots=\frac{1}{z}\left(-1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots\right)+\ldots$$

$$=\frac{1}{z}\left[-\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\ldots\right)\right]+\ldots=\frac{1}{z}\left(-e^{-1}\right)+\ldots$$

$\therefore$ *Coefficient of* $\frac{1}{z}$ *is* $-e^{-1}$

Hence Residue of $f(z)$ *at* $z = 0$ *is* $-e^{-1}$

$\therefore$ *Option (c) is correct.*

Example 30. *Consider the function* $F(z)=\int_1^2 \frac{1}{(x-z)^2}dz,$ *Im(z) >0. Then there is a meromorphic function G(z) on* $\mathbb{C}$ *that agrees with F(z) when Im (z) > 0, such that* ***(CSIR UGC NET-2016)***

(a) 1, ∞ are poles of G(z)
(b) 0,1, ∞ are poles of G(z)
(c) 1,2 are poles of G(z)
(d) 1,2 are simple poles of G(z).

Solution: (c,d) *Given* $F(z)=\int_1^2 \frac{1}{(x-z)^2}dx, \operatorname{Im} z>0$

$$=\left.\frac{-1}{(x-z)}\right|_1^2=\frac{-1}{2-z}+\frac{1}{(1-z)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$$

Given; F(z)=G(z) on Imz > 0

We know, if two functions agree on a connected domain, then they agree for all values of z.

$\therefore G(z)=F(z) \;\; \forall z$

Clearly 1, 2 are simple poles of F(z) and hence of G(z)

$\therefore$ *Options (c) and (d) are correct*

Example 31. *Consider the polynomial* $P(z)=\sum_{n=1}^{N} a_n z^n, 1\le N<\infty,\ a_n\in \mathbb{R}\backslash\{0\}$*. Then, with* $D=\{w\in\mathbb{C}: |w|<1\}$

(a) $P(D)\subseteq\mathbb{R}$ *(b) P(D) is open* *(c) P(D) is closed* *(d) P(D) is bounded*

Solution: (b,d) $P(z)=\sum_{n=1}^{N} a_n z^n\, 1\le N<\infty,\ a_n\in\mathbb{R}\setminus\{0\}$ *and* $D=\{w\in\mathbb{C}: |w|<1\}$

$\Rightarrow$ *P(z) is a polynomial function and D is open. By Open Mapping theorem, P(D) is open* $\Rightarrow$ *option (b) is correct and (c) is incorrect.*

For option (a),

Take P(z) = z ; $(a_1=1)$*, N = 1*

For w = i/2

P(z) = i/2

$\Rightarrow P(D)\nsubseteq\mathbb{R}$

$\therefore$ *Option (a) is incorrect.*

$P(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n$ *as* $a_n\in\mathbb{R}\backslash\{0\}$

$\Rightarrow$ *As Domain is bounded and P(z) is entire*

$\Rightarrow$ *P(D) is bounded*

$\therefore$ *Option (d) is correct*

Hence options (b) and (d) are correct.

Example 32. *Let D be the open unit disc in $\mathbb{C}$. Let g: D →D be holomorphic, g(0)=0, and let*

$$h(z)=\begin{cases}\dfrac{g(z)}{z} & , z\in D, z\neq 0\\ g'(0) & , z=0\end{cases}.$$ *Which of the following statements are true?*

(CSIR UGC NET DEC-2016)

(a) h is holomorphic in D.

(b) $h(D)\subseteq \overline{D}$.

(c) $|g'(1/2)|\leq 1/2$.

(d) $\left|g\left(\frac{1}{2}\right)\right|\leq\frac{1}{2}$

Solution: (a,b,d) *Since, g:D⟶D is holomorphic and g(0)=0*

∴ *By Schwarz Theorem,* $|g(z)|\leq |z|\ \forall\ z\in D$ *and* $|g'(0)|\leq 1$

In particular, $\left|g\left(\frac{1}{2}\right)\right|\leq\frac{1}{2}$

Thus, option (d) is correct and (c) is incorrect

Further, $h(z)=\begin{cases}g(z)/z & , z\in D,\ z\neq 0\\ g'(0) & , z=0\end{cases}$

$$=\begin{cases}\dfrac{g(z)-g(0)}{z-0}, & z\in D,\ z\neq 0\\ g'(0) & , z=0\end{cases}$$

Clearly, $\lim\limits_{z\to 0}h(z)=\lim\limits_{z\to 0}\dfrac{g(z)-g(0)}{z-0}=g'(z)$ *and h(0)=* $g'(0)$

$\Rightarrow h(z)=g'(z)\ \forall\ z\in D$

We know that in $\mathbb{C}$ if a function is differentiable once, then it is infinitely many times differentiable.

∴ *h(z) is holomorphic (analytic) in D.*

Thus, option (a) is correct

Further, as |g(z)| ≤ |z|

$\Rightarrow \left|\dfrac{g(z)}{z}\right|\leq 1$, *i.e., |h(z)| ≤ 1 ∀ z∈D* $\Rightarrow h(D)\subseteq\overline{D}$

∴ *option (b) is correct*

ASSIGNMENT - 5.1

NOTE: CHOOSE THE BEST OPTION

1. *The poles of first order are known as*
   (a) complex poles (b) simple poles
   (c) singularities (d) none of these

2. *A point $z = z_0$ is a singular point of analytic function $f(z)$, if*
   (a) at $z = z_0$, $f(z)$ is not analytic (b) at $z = z_0$, $f(z)$ is analytic
   (c) at $z = z_0$, $f(z) = 0$ (d) none of these

3. *The function $f(z) = z^2$ have zero of order*
   (a) one (b) two
   (c) three (d) four

4. *The function $f(z) = \cos z$ have zero of order*
   (a) one (b) two
   (c) ∞ (d) none of these

5. *If $f(z)$ is analytic and has a pole at $z = z_0$, then*
   (a) $|f(z)| = C$, a constant as $z \to z_0$ (b) $|f(z)| \to 0$, as $z \to z_0$
   (c) $|f(z)| \to \infty$, as $z \to z_0$ (d) None of these

6. *The zero of first order is known as*
   (a) complex zero (b) simple zero
   (c) singularity (d) none of these

7. *The function $f(z) = z$ has isolated singularity at*
   (a) ∞ (b) 0
   (c) $1/\infty$ (d) none of these

8. *If f has a pole of order m at $z = a$ and $g(z) = (z-a)^m f(z)$, then*
   (a) $Res\,(f;\,a) = \dfrac{1}{(m-1)!} g^{(m-1)}(a)$ (b) $Res\,(f;\,a) = g^{(m-1)}(a)$
   (c) $Res\,(f;\,a) = \dfrac{1}{(m-1)!} g(a)$ (d) none of these

9. *If $z = a$ is an isolated singularity of f, then a is the pole of f, if*
   (a) $\lim\limits_{z\to a} |f(z)| = 0$ (b) $\lim\limits_{z\to a} |f(z)| = a$
   (c) $\lim\limits_{z\to a} |f(z)| = \infty$ (d) none of these

10. Let a be an isolated singularity of $f(z)$ and if $|f(z)|$ is bounded on some neighbourhood of a, then a is
(a) removable singularity
(b) essential singularity
(c) pole
(d) non – isolated singularity

11. Polynomial of degree n has a pole of order n at
(a) zero
(b) infinity
(c) curve C: $|z| = 1$
(d) anywhere

12. The function $f(z) = \frac{1}{z^n}$ is/has
(a) analytic for all z
(b) singularity at z=0
(c) singularity at n=0
(d) none of these

13. If G is a region and f is non-constant analytic function on G. Then, open mapping theorem states, for any open set $U \subset G$
(a) $f(U)$ is closed
(b) $f(U)$ is open
(c) $f(U) = U$
(d) none of these

14. The function tan z have singularities at
(a) $\left(k + \frac{1}{2}\right)\pi$, $k = 0, \pm 1, \pm 2, \ldots$
(b) $\frac{1}{2}k\pi$
(c) $2k\pi$
(d) none of these

15. The function f(z)=1/z has isolated singularity at
(a) ∞
(b) 0
(c) 1/0
(d) None of these

16. If f(z) has second order zero at $z = z_0$, then
(a) $f(z_0) = f'(z_0) = 0$ and $f''(z_0) \neq 0$
(b) $f(z_0) = f'(z_0) = 0$ and $f'(z_0) \neq 0$
(c) $f(z_0) = f'(z_0) = f''(z_0) = 0$
(d) none of these

17. The origin is the zero of $z^3 \sin z$ of order
(a) 1
(b) 3
(c) 4
(d) none of these

18. If f has an isolated singularity at z = a and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion about z = a. Then residue of f at z = a is
(a) a_{-1}
(b) a_0
(c) a_{-2}
(d) a_1

19. $\int_c \frac{e^{3z}}{(z-\pi i)}dz$, where C is circle $|z-1|=6$, is

(a) $-2\pi i$ (b) 0

(c) πi (d) $2\pi i$

20. If $z = a$ is an isolated singularity of f and $f(z)=\sum_{-\infty}^{\infty} a_n(z-a)^n$ is its Laurent Expansion in an $(a; 0, R)$. Then $z = a$ is a removable singularity, if

(a) $a_n = 0, n \leq -1$ (b) $a_n \neq 0, n \leq -1$

(c) $a_n = 0, n \geq -1$ (d) $a_n \neq 0, n \geq -1$

21. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ is its Laurent expansion in $(a; 0, R)$.

Then $z = a$ is a pole of order m, if

(a) $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m + 1)$

(b) $a_{-m} = 0$ and $a_n \neq 0$ for $n \leq -(m + 1)$

(c) $a_{-m} = 0$ and $a_n = 0$ for $n \leq -(m + 1)$

(d) None of these

22. If $z = a$ is an isolated singularity of f and $f(z)=\sum a_n(z-a)^n$ is its Laurent expansion in $(a; 0, R)$. Then $z = a$ is an essential singularity if

(a) $a_n \neq 0$ for all integers n

(b) $a_n = 0$ for all integers n

(c) $a_n \neq 0$ for infinitely many negative integers n

(d) $a_n \neq 0$ for infinitely many positive integers n

23. $f(z) = \log(z + 2)$ has branch point at

(a) $z = -2$ (b) $z = \pm 2$

(c) $z = 0$ (d) $z = \infty$

24. For the function $f(z) = \frac{1-e^z}{z}$, the point $z=0$ is

(a) an essential singularity (b) a pole of order zero

(c) a pole of order one (d) a removable singularity

25. In the finite z plane if $f(z) = \frac{\cos z}{(z+i)^3}$, then

(a) $z = -i$ pole of order 3 (b) $z = +i$ pole of order 3

(c) $z = 1$ pole of order 2 (d) $z = -1$ pole of order 1

26. *The function* $f(z) = \tan z$

(a) *has poles at* $z = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
(b) *is an entire function*
(c) *has no zeros in* $\mathbb{C}$
(d) *has a removable singularity at* $z = 0$

27. *Let* $f(z) = \frac{\sin(z-1)}{z-1}$, *then*

(a) *f(z) has simple pole at z=1*
(b) *f(z) has essential singularity at z=1*
(c) *f(z) has removable singularity at z=1*
(d) *residue of f(z) at z=0 is undefined*

28. *For the function* $f(z) = \frac{\log_e(z-2)}{(z^2+2z+2)^4}, z = -1-i$ *is a pole of order*

(a) 1
(b) 2
(c) 3
(d) 4

29. *The number of isolated singular points of* $f(z) = \frac{z+3}{z^2(z^2+2)}$ *is*

(a) 1
(b) 2
(c) 3
(d) 4

30. *The value of* $\int_C \frac{zdz}{\sin z}$, *where C: |z|=4, is*

(a) $2\pi i$
(b) 0
(c) $-2\pi i$
(d) $4\pi i$

31. *The poles of the function* $f(z) = \frac{\sin z}{\cos z}$ *are at*

(a) $\frac{(2n+1)\pi}{2}, n$ *is an integer*
(b) $\frac{2n\pi}{3}, n$ *is any integer*
(c) $n\pi$
(d) *none of these*

32. *Which of the following(s) is/are correct?*

(a) *the function f(z) has a pole of order m at the point z=a, then*

$$\text{Res } f(a) = \frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]$$

(b) *the residue of f(z) at infinity is* $\lim_{z\to\infty}[zf(z)]$

(c) *the residue of* $f(z) = \frac{z}{(z-1)(z-2)}$ *at* $z = \infty$ *is 1*

(d) *none of the above*

33. *The function sin z is analytic in*

(a) $\mathbb{C} \cup \{\infty\}$
(b) $\mathbb{C}$
(c) $\mathbb{C} - \{0\}$
(d) $\mathbb{C}$ *except on negative real axis*

34. *A point z_0 is called zero of $f(z)$, if*

(a) $f(z_0)$ is constant (b) $f(z_0) = 0$ (c) $f(z_0) \geq 0$ (d) none of these

NOTE : MORE THAN ONE OPTION MAY BE CORRECT

35. Let $f(z)=\dfrac{z}{(z+1)(z+2)}$, then $f(z)$ has

(a) simple pole at $z=-1$
(b) simple pole at $z=-2$
(c) simple pole at $z=-1$ and $z=-2$
(d) pole of order 2.

36. If $f(z)$ is entire, then

(a) $f(z)$ is analytic for all z
(b) $f(z)$ is differentiable for all z
(c) $f(z)$ is not analytic for all z
(d) $f(z)$ is continuous for all z

37. If f is an entire function, then

(a) f has a power series expansion
(b) f has no power series expansion
(c) f is not necessarily constant
(d) none of these

38. If $f(z) = z^3$, then it

(a) has an essential singularity at $z = \infty$
(b) has a pole of order 3 at $z = \infty$
(c) has not a pole of order 3 at $z = 0$
(d) is analytic at $z = \infty$

39. The function $f(z)=\dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$ have the poles at

(a) $z = 1$
(b) $z = 2$
(c) $z = -1$
(d) $z = -2$

40. Which of the following is/are true?

(a) Zeros of an analytic function are isolated
(b) Poles are isolated
(c) A zero of order two is called simple zero
(d) The value of z for which the function $f(z)$ fails to be analytic is called zero

41. Which of the following gives the residue at $z= \infty$?

(a) $-\text{Res}_{z=0} \dfrac{1}{z^2} f\left(\dfrac{1}{z}\right)$

(b) $-\dfrac{1}{2\pi i}\int_C f(z)dz$

(c) Negative of the coefficient of $\dfrac{1}{z}$ in the expansion of $zf(z)$ in a neighbourhood of $z = 0$

(d) all of above

42. Which of the following(s) is/are true ?
(a) The limit point of sequence of zeros is non isolated essential singularity
(b) The limit point of sequence of poles is non isolated essential singularity
(c) The limit point of sequence of poles is isolated as well as non isolated essential singularity
(d) The limit point of sequence of zeros is an isolated essential singularity.

43. The branch point of $f(z) = (z^2 + 1)^{1/2}$ is/are
(a) i
(b) $-i$
(c) 1
(d) $f(z)$ is single valued

44. Residue of $\dfrac{z^3}{z^2-1}$ is
(a) -1 at $z=\infty$
(b) $1/2$ at $z=1$
(c) 0 at $z=\infty$
(d) $1/2$ at $z=-1$

45. Let $f(z) = \dfrac{z^2+1}{(z^3+1)(z-1)}$. The singular points of this function are
(a) 1
(b) w
(c) w^2
(d) -1

46. The pole of the function $f(z)=\dfrac{z^2}{(z-1)^2(z+2)}$ is
(a) at $z=-2, 1$
(b) at $z=-1$
(c) at $z=+1$
(d) at $z=+2$

47. Which of the following(s) is/are correct?
(a) Pole is sometimes known as an essential singularity
(b) If $f(z)=e^{1/z}$, then $z=0$ is a point of essential singularity
(c) The function $f(z)=\tan(1/z)$ has a removable singularity at $z=0$
(d) None of the above

48. Which of the following is/are correct ?
(a) If a single valued function $f(z)$ is not defined at $z=a$, but $\lim\limits_{z\to a} f(z)$ exists, then $z=a$ is known as a removable singularity
(b) If for the function $f(z)=\dfrac{\sin z}{z}$, $f(0)$ is defined as 2, then $z=0$ is a removable singularity
(c) Every polynomial of degree n has exactly n roots in $\mathbb{C}$
(d) None of the above

49. Consider $P(z)=z^{10}-6z^7+3z^3+1$ and $|z|<1$, then
(a) $P(z)$ is analytic on $|z|=1$
(b) $P(z)$ has six zeros inside $|z|=1$
(c) $P(z)$ has seven zeros inside $|z|=1$
(d) Only (b) is true

50. Let f(z) be a complex valued function. Then which of the following is/are true?
   (a) The order of a zero of polynomial equals the order of its first non vanishing derivative
   (b) Zeros are isolated
   (c) Zeros of f(z) are obtained by equating numerator to zero
   (d) Limit point of zeros is an essential singularity

51. For the function $f(z)=\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$. Which of the following holds.
   (a) z=1 is the pole of order 4
   (b) z=2 is the simple pole
   (c) z=0 is the zero of order 3
   (d) z=1,2,3 are the simple poles

52. Let $f(z)=\dfrac{1}{(z^2+9)^2}$, then
   (a) $z=3i$ is a simple pole of f(z)
   (b) $z=-3i$ is a pole of order two of f(z)
   (c) f(z) has removable singularity at $z=3i$
   (d) Residue of f(z) at $z=3i$ is $=\dfrac{-i}{108}$

53. Which of the following is/are correct?
   (a) The zeros of an analytic function f(z) are isolated
   (b) A singularity of a function f(z) is a value of z at which the function f(z) ceases to be analytic
   (c) A polynomial of nth degree has a pole of nth order at infinity
   (d) None of the above

54. Given $f(z)=\dfrac{z^n}{(z-1)^2(z+2)}$, then
   (a) the function has simple pole at z=-2
   (b) the function has no pole
   (c) the residue of the function at z= -2 is 4/9
   (d) the function have no residue

55. Given $f(z)=\oint_C \dfrac{z^2-z+1}{z+1}dz$, where C is the circle $|z|=\dfrac{1}{4}$,then
   (a) f(z) is analytic everywhere within C.
   (b) f(z) is not analytic
   (c) f(z)=0
   (d) f(z) has a simple pole

56. Given $f(z)=\oint_C g(z)dz$, where $g(z)=\dfrac{z-3}{z^2-2z+5}$ and C is a circle $|z|=1$, then
   (a) f is analytic everywhere within C.
   (b) f(z)=0 within C.
   (c) the poles of g(z) lies outside the circle
   (d) the poles of g(z) lies inside the circle

57. Let $f(z)=e^z$, then which of the following hold?
   (a) f(z) is an entire function
   (b) f(z) has an isolated essential singularity at z=0
   (c) f(z) has an isolated essential singularity at $z=\infty$
   (d) f(z) has removable singularity at z=0.

ASSIGNMENT - 5.2

NOTE: CHOOSE THE BEST OPTION

1. *The residue of the function* $f(z) = \dfrac{z^2}{(z-1)^2(z-2)}$ *(at z = 2), is*

   (a) 4 (b) 2 (c) 1 (d) none of these

2. *The function* $f(z) = z^m e^z$ *at* $z = \infty$ *, where m is a natural number has*

   (a) non – isolated essential singularity (b) pole of order m
   (c) pole of order m^2 (d) isolated essential singularity

3. *The residue of the function* $f(z) = \dfrac{2z}{(z+4)(z-1)^2}$ *at the point z= 1, is*

   (a) $\dfrac{1}{5}$ (b) $\dfrac{2}{5}$ (c) $\dfrac{8}{25}$ (d) $\dfrac{4}{25}$

4. $f(z) = \dfrac{\sin z}{(z-\pi)^2}$ *has a pole of order*

   (a) 1 (b) 2 (c) 3 (d) 0

5. *For the function* $f(z)=\dfrac{z-\sin z}{z^5}$*, the point z=0 is*

   (a) a pole of order 3 (b) a pole of order 2
   (c) an essential singularity (d) an removable singularity

6. $f(z) = \dfrac{z+2}{z(z^2+1)}$ *has singularity at*

   (a) 0, 1 (b) 0, i (c) 0, $\pm i$ (d) 0, 1, i, $-i$, 2

7. *The function sin z has simple zeros at*

   (a) $z = 0$ only (b) $z = 0, \pm 2\pi$ only (c) $z = 0, \pm\pi, \pm 2\pi, \ldots$ (d) none of these

8. *The function* e^z *has*

   (a) isolated essential singularity at 0 (b) singularity at infinity
   (c) essential singularity at infinity (d) none of these

9. *Let F be any circle enclosing the origin and oriented counter – clockwise, then the value of the integral* $\displaystyle\int_F \frac{\cos z}{z^2}\,dz$ *is*

   (a) $2\pi i$ (b) 0 (c) $-2\pi i$ (d) undefined

10. Residue of $\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at double pole at $z=-1$ is

(a) 4/5 (b)-4/5 (c) -14/25 (d) 14/25

11. Residue of $f(z)=\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z=3$, is

(a) $\frac{101}{16}$ (b) -8 (c) $\frac{27}{16}$ (d) 0

12. Let Γ denotes the boundary of the square whose sides lie along $x=\pm 1$ and $y=\pm 1$, where Γ is described in the positive sense. Then the value of $\int_{\Gamma}\frac{z^2}{2z+3}dz$ is

(a) $\frac{\pi i}{4}$ (b) $2\pi i$ (c) 0 (d) $-2\pi i$

13. Let $I=\int_C \frac{\cot(\pi z)}{(z-i)^2}dz$, where C is contour $4x^2+y^2=2$ (counter clockwise). Then I is equal to

(a) 0 (b) $-2\pi i$ (c) $2\pi i\left(\frac{\pi}{\sinh^2 \pi}-\frac{1}{\pi}\right)$ (d) $\frac{-2\pi^2 i}{\sinh^2 \pi}$

14. For the function $f(z)=\frac{e^{2z}}{(z-1)^3}$, the point $z=1$ is

(a) removable singularity (b) pole of order 1
(c) pole of order 3 (d) essential singularity

15. Let $f(z)=\frac{z^2}{(z-a)(z-b)(z-c)}$, then

(a) $\lim_{z\to\infty} f(z)=1$ (b) $\lim_{z\to\infty} zf(z)=-1$
(c) residue of $f(z)$ at infinity $=1$ (d) $\lim_{z\to\infty} -zf(z)=-1$

16. For the function $f(z)=\frac{\pi\cos\pi z}{z^2\sin\pi z}$, which of the following is true?

(a) $z=1,2,3,\ldots$ are the only simple poles (b) $z=0$ is pole of order 2
(c) $z=0$ is pole of order 3 (d) Residue at $z=0$ is $\frac{\pi^2}{3}$

17. The residue of $f(z)$ at $z=2$ where $f(z)=\frac{e^{-z}}{(z-2)^4}$, is

(a) $\frac{1}{6}$ (b) $\frac{e^2}{6}$ (c) $-\frac{1}{6e^2}$ (d) $\frac{1}{6e^2}$

18. *For the function* $f(z)=\sin\left(\frac{x}{|z|^2}-i\frac{y}{|z|^2}\right)$. *Choose the correct option.*

(a) *f(z) has no singularity* (b) *f(z) has exactly one singularity*
(c) *all the singularities of f(z) are poles* (d) *infinity is simple pole*

19. *If* $f(z)=z^5-3iz^2+2z+i-1$ *and C encloses zeros of f(z), then* $\int_C \frac{f'(z)}{f(z)}dz$ *is*

(a) $5\pi i$ (b) 0 (c) $10\pi i$ (d) *None of these*

20. *Number of poles of the function* $f(z)=\tan\frac{1}{z}$ *is*

(a) 2 (b) 4 (c) *infinite* (d) *none of these*

21. *The sum of the residues of* $f(z)=\frac{\sin z}{z\cos z}$ *at its poles inside the circle* $|z|=2$, *is*

(a) 0 (b) 1 (c) -1 (d) π

22. *The zeros of the function* $\sin(3iz+1)$ *are*

(a) *zero* (b) $i(n\pi+1)/2$ (c) $i(n\pi+1)/3$ (d) *none of these*

NOTE : MORE THAN ONE OPTION MAY BE CORRECT

23. $f(z)=\sin(1/z)$, $z=0$ *is a/an*

(a) *removable singularity* (b) *simple pole* (c) *isolated singularity* (d) *essential singularity*

24. *The residue of* $\frac{\cos z}{z}$ *is*

(a) -1 *at* $z=\infty$ (b) 1 *at* $z=0$ (c) -1 *at* $z=0$ (d) 1 *at* $z=\infty$

25. *At* $z=0$ *the function* $f(z)=\frac{1-\cos z}{z}$ *has not*

(a) *a simple pole* (b) *a removable singularity*
(c) *an essential singularity* (d) *a non-isolated singularity*

26. $\int_{|z|=1} e^{1/z}dz$ *is not equal to*

(a) πi (b) $2\pi i$ (c) $-2\pi i$ (d) $-\pi i$

27. *The residue of* $\frac{\sin^2 z}{z^8}$ *at* $z=0$ *is not*

(a) 0 (b) $-1/7!$ (c) $1/7!$ (d) *none of these*

28. $f(z)=2z+6z^3$ *has not*

(a) *a pole of second order at infinity* (b) *a pole of third order at infinity*
(c) *a pole of fourth order at infinity* (d) *all of these*

29. Let $f(z)=\frac{z}{(z-1)(z-3)}$, C is the circle $|z|=\frac{3}{2}$. Then $\oint_C f(z)\,dz$ equals to

(a) $2\pi i[f(1)(z-1)+f(3)(z-3)]$, by Cauchy's integral formula
(b) $2\pi i[f(1)(z-1)]$, by Cauchy's integral formula
(c) $2\pi i[\text{Res}(z=1)+\text{Res}(z=3)]$, by Residue Theorem formula
(d) $2\pi i[\text{Res}(z=1)]$, by Residue Theorem formula

30. For $f(z) = z \operatorname{cosec} z$, which of the following(s) is/are true?

(a) The poles of $f(z)$ are $z=n\pi, n\in\mathbb{Z}, n\neq 0$
(b) $z=\infty$ is a non isolated essential singularity
(c) The poles of $f(z)$ are at $z=0$
(d) $z=\infty$ is the limit point of the poles

31. For $f(z)=\frac{z-2}{z^2}\sin\frac{1}{z-1}$, which of the following(s) is/are true?

(a) $z=0$ is a pole of order one
(b) $z=0$ is a pole of order two
(c) $z=1$ is an isolated essential singularity of $f(z)$
(d) $z=1$ is a limit point of the zeros

32. Which of the following is/are correct?

(a) The zeros of an analytic function $f(z)$ are isolated.
(b) The residue of $f(z)$ at infinity $=\lim_{z\to\infty}[zf(z)]$.
(c) An integral function is the function which is analytic everywhere.
(d) None of the above.

33. If R_1, R_2, R_3 and R_∞ denotes the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and ∞ respectively, then

(a) $R_1+R_2-R_3+R_\infty=0$
(b) $R_1+R_2+R_3-R_\infty=0$
(c) $R_1-R_2+R_3+R_\infty=0$
(d) $R_1+R_2+R_3+R_\infty=0$

34. Which of the following is/are correct?

(a) The expansion of $\frac{1}{(z^2-3z+2)}$ in a Laurent series for $0<|z|<1$ is $\sum_{n=0}^{\infty}\left(\frac{z^n}{2^{n+1}}\right)$.
(b) If $f(z)$ has a zero of order n at $z=a$, then $\frac{1}{f(z)}$ has a pole of order n at $z=a$.
(c) The function $f(z)$ is said to be singular at $z=0$ if $f(1/z)$ is singular at $z=0$.
(d) None of these

35. Laurent expansion of $f(z)$ is given by $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$, where

(a) $a_n=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{(z-z_0)^{n+1}}dz$
(b) $b_n=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{(z-z_0)^{n+1}}dz$
(c) $\text{Res}\, f(z_0)=b_1$
(d) z_0 is removable singularity for all $b_n=0$

ASSIGNMENT - 5.3

NOTE: CHOOSE THE BEST OPTION

1. *The function g, defined by* $g(z)=z^4\sin\left(\frac{1}{z+1}\right)$ *has at infinity*

   (a) a pole of order 4 (b) a pole of order 3
   (c) removable singularity (d) essential singularity

2. *The value of the integral* $I=\int_c \frac{e^z}{e^z-1}dz$, *where* $C:|z|=3\pi$, *is*

   (a) $2\pi i$ (b) $3\pi i$ (c) $6\pi i$ (d) 0

3. *The residue at* $z=1$ *for* $f(z)=\frac{\cos z}{z-1}$, *is*

   (a) e (b) e^i (c) $\frac{e^i+e^{-i}}{2}$ (d) $\frac{e^i-e^{-i}}{2i}$

4. *If* $f(z)=\frac{1}{\sinh z}$ *then the number of singularities of f(z) on the line segment* $(-n\pi, n\pi)$ *is*

   (a) n (b) 2n-1 (c) 2n+1 (d) 1

5. *For the function* $f(z)=e^{\frac{1}{z^4}}$, *at z = 0*

   (a) limit does not exist (b) limit exists but $f(z)$ is discontinuous
   (c) $f(z)$ is continuous but not differentiable (d) f(z) is differentiable

6. *If* $I=\int_{|z|=2.5} z^2\cot\pi z\,dz$, *then I is equal to*

   (a) $20\pi i$ (b) $20i$ (c) $20\pi^2 i$ (d) $10i$

7. *The number of singularities of coth (z/2) in the circle |z| =12 is*

   (a) zero (b) one (c) three (d) four

8. *Let* $n\in\mathbb{Z}$. *The set of all singularities of the function* $f(z)=z+|z|$ *is*

   (a) Only $n\pi$ (b) only $2n\pi$ (c) $\mathbb{R}$ (d) ϕ

9. *Consider the function* $\frac{\log(z-i)}{z^2+4}$. *Its singularities are*

   (a) $\{0,2i,-2i\}$ (b) $\{2i,-2i, y=-1 (x\le 0)\}$
   (c) $\{1,2i,-2i\}$ (d) $\{2i,-2i, y=1(x\le 0)\}$

10. *The singular points of cosech z Log z are*
(a) $z=2n\pi$
(b) $z=n\pi i$
(c) $z=(2n+1)$
(d) $z=(2n+1)\pi i$

11. *The number of singularities of tanh z in the circle* $|z| = 12$ *is*
(a) 1
(b) 2
(c) 4
(d) none of these

12. *The singularities of the function* $f(z)=\dfrac{e^z}{\sin z-\sin a}$, $a\neq\left(n+\dfrac{1}{2}\right)\pi$, $n\in\mathbb{Z}$ *are the simple poles at*
(a) $n\pi + a$
(b) $n\pi + (-1)^n a$
(c) $n\pi - a$
(d) none of these

13. *Let f(z) be an even analytic function having a pole of order two at* $z = a$ *. Then the residue of f(z) at* $z = a$ *is*
(a) -1
(b) 0
(c) 1
(d) 2

14. *The residue of the function* $f(z) = \dfrac{\sin z^2}{z^5 \sin z}$ *at* $z = 0$, *is*
(a) -1
(b) 0
(c) 1
(d) 2

15. *The residue of tan z at* $z = \dfrac{\pi}{2}$, *is*
(a) -1
(b) 1
(c) 0
(d) 2

16. *The residue of tanh z at* $z = \dfrac{\pi i}{2}$, *is*
(a) -1
(b) 1
(c) 0
(d) 2

17. *The residue of* $\pi \operatorname{cosec}(\pi z)$ *at* $z = n, n\in\mathbb{Z}$, *is*
(a) $(-1)^n$
(b) n
(c) 0
(d) $-n$

18. *The number of roots of the equation* $z^5-12z^2+14=0$ *that lie in the region* $\left\{z\in\mathbb{C}: 2\leq|z|<\dfrac{5}{2}\right\}$ *is*
(a) 2
(b) 3
(c) 4
(d) 5

19. *The general term in the expansion of coshz with the help of Taylor series is*

(a) $\frac{z^{2n+1}}{(2n+1)!}$ (b) $\frac{z^{2n}}{(2n)!}$

(c) $\frac{z^n}{n!}$ (d) None of these

20. *The function $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \sin z$ is*

(a) one-to-one (b) onto

(c) one-to-one and onto (d) neither one-to-one nor onto

21. *The value of* $\int_{|z|=2} \frac{e^{2z}}{(z+1)^4} dz$ *is*

(a) $2\pi i\, e^{-1}$ (b) $\frac{8\pi i}{3} e^{-2}$

(c) $\frac{2\pi i}{3} e^{-2}$ (d) 0

22. *The residue of $f(z) = \cot z$ at any of its poles is*

(a) 0 (b) 1

(c) $\sqrt{3}$ (d) none of these

23. *The number of zeros, counting multiplicities, of the polynomial $z^5 + 3z^3 + z^2 + 1$ inside the circle $|z| = 2$, is*

(a) 0 (b) 2

(c) 3 (d) 5

24. *Let γ be a simple closed curve in the complex plane Then the set of all possible values of* $\oint_\gamma \frac{dz}{z(1-z^2)}$ *is*

(a) $\{0, \pm \pi i\}$ (b) $\{0, \pi i, 2\pi i\}$

(c) $\{0, \pm \pi i, \pm 2\pi i\}$ (d) $\{0\}$

25. *The singularity of $e^{\sin z}$ at $z=\infty$ is*

(a) a pole (b) a removable singularity

(c) a non-isolated essential singularity (d) an isolated essential singularity

26. *Let f and g be analytic in a domain D and let each have zeros of order m and n at $z = z_0$. Then order of zeros of fg at $z = z_0$ is*

(a) $m + n$ (b) $\leq \min\{m, n\}$

(c) $\leq \max\{m, n\}$ (d) mn

NOTE : MORE THAN ONE OPTION MAY BE CORRECT

27. *The function* $f(z)=\sin\frac{1}{z}$ *has*
   (a) removable singularity at $z = 0$
   (b) simple pole at $z = 0$
   (c) no essential singularity at $z=\infty$
   (d) essential singularity at $z=0$

28. *If* $f(z) = u + iv$ *is analytic and (u, v) lies on a circle of unit radius with center at origin for all values* $z\in \mathbb{C}$, *then which of the following(s) is not true?*
   (a) f(z) has countably infinite singularities
   (b) f(z) is non – constant entire function
   (c) f(z) is meromorphic function with non empty set of singularity
   (d) f (z) is constant

29. *If* $f(z) = x^2y^2 + 2ix^2y^2$, *then which of the following statement(s) is/are correct?*
   (a) f(z) is differentiable at infinite number of points
   (b) f(z) has finite number of singularities
   (c) f(z) is nowhere analytic
   (d) f(z) is analytic everywhere

30. *Which of the following is false for a non constant entire function?*
   (a) cannot have uncountable number of zeros in $\mathbb{C}$.
   (b) can have a countable number of zeros in a bounded region of $\mathbb{C}$.
   (c) cannot have three zeros lying on a straight line.
   (d) should have at least one zero in $\mathbb{C}$.

31. *The singularity of the function* $\dfrac{\log(2z+3)}{(2z+3)(z+2)}$ *is at*
   (a) -2
   (b) $-\dfrac{3}{2}$
   (c) $y=0,\left(x\le-\dfrac{3}{2}\right)$
   (d) 2

32. *If* $f(z)=z^3-z+1$, *then* $\dfrac{1}{2\pi i}\displaystyle\int_C z^2\frac{f'(z)}{f(z)}dz$ *is not equal*
   (a) -3
   (b) -5
   (c) 2
   (d) 0

33. *If* $f(z) =\dfrac{1-e^z}{1+e^z}$, *then at* $z = \infty$, $f(z)$ *has*
   (a) pole
   (b) essential singularity
   (c) isolated singularity
   (d) non–isolated singularity

34. *For the function* $f(z) = \dfrac{\cot \pi z}{(z-a)^2}, a \notin \mathbb{Z}$, *which of the following(s) is/are true?*

(a) $z = n$, $(n \in I)$ *are simple poles of f(z) if n is finite*

(b) $z = \infty$ *is the non-isolated essential singularity*

(c) *Residue at z=n is* $\dfrac{1}{\pi(n-a)^2}$

(d) $z = a$ *is the pole of order two*

35. *Which of the following(s) is/are not correct?*

(a) *If the function f(z) has a simple pole at z=a, then* Res $f(a) = \lim_{z \to a}[(z-a)f(z)]$

(b) *If the function* $f(z) = \dfrac{F(z)}{G(z)}$ *has a simple pole at z=a, then* Res $f(a) = \dfrac{F'(a)}{G(a)}$

(c) *The function f(z)=tan(1/z) has a removable singularity at z=0*

(d) *None of these*

36. *Let* $f(z) = e^z \operatorname{cosec}^2 z$, *then for f(z)*

(a) *z=0 is the pole of order 2*

(b) $z = \infty$ *is a non isolated essential singularity*

(c) $z = n\pi, n \in \mathbb{Z}$ *are the poles of order 2*

(d) *Residue at z=0 is 0*

37. *The value of the integral* $\oint_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-4)(z-2)} dz$, *where C is the circle* $|z| = 3$ *traced anti clockwise, is*

(a) $-2i\pi$

(b) $i\pi$

(c) $-i\pi$

(d) $2i\pi$

38. *The value of the integral* $\oint_C \dfrac{dz}{z^2-1}$, $C : |z| = 4$ *is equal to*

(a) πi

(b) 0

(c) $-\pi i$

(d) $2\pi i$

39. *Which of the following(s). is/are true?*

(a) *A polynomial of degree n has no singularities in the finite part of the plane but has a pole of order n at infinity*

(b) *A function which has no singularity in the finite part of the plane or at infinity is constant*

(c) *A function f(z) is a polynomial of degree n if and only if f(z) has no singularities in the finite part of the plane and has a pole of order n at infinity.*

(d) *If a function f(z) is analytic for all finite values of z and as* $|z| \to \infty$, $|f(z)| = a|z|^k$, *then f(z) is a polynomial of degree* $\leq k$.

ANSWERS TO EXERCISES

PRACTICE SET – I

Exercise 1: (a) *Exercise 2: (c)* *Exercise 3: 0.041* *Exercise 4: (b,c,d)*
Exercise 5: (d) *Exercise 6: (a,c)* *Exercise 7: (a)*

PRACTICE SET – II

Exercise 4: (a,b,c) *Exercise 5: (c)*

ANSWERS TO ASSIGNMENTS

ASSIGNMENT - 5.1

1. (b)	2. (a)	3. (b)	4. (a)	5. (c)	6. (b)	7. (a)
8. (a)	9. (c)	10. (a)	11. (b)	12. (b)	13. (b)	14. (a)
15. (b)	16. (a)	17. (c)	18. (a)	19. (a)	20. (a)	21. (a)
22. (c)	23. (a)	24. (d)	25. (a)	26. (a)	27. (c)	28. (d)
29. (c)	30. (b)	31. (a)	32. (a)	33. (b)	34. (b)	35.(a,b,c)
36. (a,b,d)	37. (a,c)	38. (b,c)	39. (a,b)	40. (a,b)	41. (a,b,c,d)	42.(b,d)
43. (a,b)	44. (a,b,d)	45. (a,d)	46. (a,c)	47. (b)	48. (a,b,c)	49.(a,c)
50. (a,b,c,d)	51. (a,b,c)	52. (b,d)	53. (a,b,c)	54. (a,c)	55. (a,c)	56. (a,b,c)
57. (a,c)						

ASSIGNMENT - 5.2

1. (a)	2. (d)	3. (c)	4. (a)	5. (b)	6. (c)	7. (c)
8. (c)	9. (b)	10. (c)	11. (c)	12. (c)	13. (c)	14. (c)
15. (d)	16. (c)	17. (c)	18. (b)	19. (c)	20. (c)	21. (a)
22. (c)	23. (c,d)	24. (a,b)	25. (a,c,d)	26. (a,c,d)	27. (b,c,d)	28. (a,c,d)
29. (b,d)	30. (a,b,d)	31. (b,c,d)	32. (a,c)	33. (d)	34. (b)	35. (a,c,d)

ASSIGNMENT - 5.3

1. (b)	2. (c)	3. (c)	4. (d)	5. (a)	6. (b)	7. (c)
8. (d)	9. (d)	10. (b)	11. (d)	12. (b)	13. (b)	14. (b)
15. (a)	16. (b)	17. (a)	18. (b)	19. (b)	20. (b)	21. (b)
22. (b)	23. (d)	24. (c)	25. (d)	26. (a)		
27. (c, d)	28. (a,b,c)	29. (a,c)	30. (b,c,d)	31. (a,b,c)	32. (a,b,d)	33. (b,d)
34. (a,b,c,d)	35. (b,c)	36. (a,b,c)	37. (c)	38. (b)	39. (a,b,c,d)	

CHAPTER - 6
EVALUATION OF DEFINITE INTEGRALS AND BILINEAR TRANSFORMATIONS

INTRODUCTION

The integrals discussed in this chapter are those termed as definite integrals. Definite integrals can be recognized by numbers written to the upper and lower of the integral sign. The standard method of evaluating a definite integral is based on 'Fundamental theorem of Calculus'. Many types of real definite integrals can be found using the results of contour integrals in complex plane. We use residues to find these integrals. The calculus of residues often provides an efficient method for evaluating certain real and complex integrals. We shall discuss this in the beginning of this chapter. Another important class of elementary functions is the mobius transformations or bilinear transformations. Mobius transformation provides a very convenient method of finding a one-to-one mapping from one domain into another. In this section, we will discuss how they are used to map a disk onto a disk or a half plane onto half plane.

6.1. CONTOUR INTEGRATION

6.1.1. Improper Integral Involving Rational Functions: *The evaluation of definite integrals is often achieved by using the residue theorem together with a suitable function f(z) and a suitable closed path or contour C. The following types are most common in practice.*

(A) $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$, *where $F(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$. Here θ is the argument of a point z on the unit circle centered at origin. Let $z = e^{i\theta}$. Then* $\sin\theta = \frac{z - z^{-1}}{2i}$,

$\cos\theta = \frac{z + z^{-1}}{2}$ *and* $dz = ie^{i\theta}\, d\theta$ *or* $d\theta = dz/iz$ [$\because$ *on* $|z|=1$, $\bar{z} = z^{-1}$]. *The given integral is equivalent to* $\int_0^{2\pi} F(\sin\theta, \cos\theta) d\theta = \int_c F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{iz}$, *where C is the positively oriented unit circle with centre at origin.*

For Example: *Show that* $\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta} = \frac{\pi}{\sqrt{2}}$.

Solution: *Let* $I = \int_0^{2\pi} \frac{d\theta}{3 + \cos\theta}$

Here, $F(\sin\theta, \cos\theta) = \frac{1}{3 + \cos\theta}$

Let $z = e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow z^{-1} = \cos\theta - i\sin\theta$

$\Rightarrow \sin\theta = \frac{z - z^{-1}}{2i}, \cos\theta = \frac{z + z^{-1}}{2}$ *and* $dz = i.e^{i\theta}\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\therefore I = \oint_{|z|=1} \frac{1}{3+\left(\frac{z+z^{-1}}{2}\right)} \times \frac{dz}{iz} = \oint_{|z|=1} \frac{2z}{6z+z^2+1} \times \frac{dz}{iz}$$

$$= \frac{2}{i}\oint_{|z|=1} \frac{dz}{(z^2+6z+1)} = \frac{2}{i}\oint_{|z|=1} \frac{dz}{(z-(-3+\sqrt{8}))(z+3+\sqrt{8})}$$

$$= \frac{2}{i} \times 2\pi i \left(\frac{1}{-3+\sqrt{8}+3+\sqrt{8}}\right) = 4\pi \times \frac{1}{2\sqrt{8}} = \frac{2\pi}{\sqrt{8}} = \frac{\pi}{\sqrt{2}}$$

(B) $\int_{-\infty}^{\infty} F(x)\,dx$, *F(x) is a rational function of the real variable x. Consider* $\oint_C F(z)\,dz$ *along a contour C consisting of the semi circle C, |z|=R above the x axis having the line segment of the real axis from –R to +R as diameter. Then, let* $R \to \infty$, *if F(x) is an even function, then* $\oint_C F(z)\,dz$ *can be used to evaluate* $\int_0^{\infty} F(x)\,dx$

For Example: *(i)* $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4.$ *(ii)* $\int_0^{\infty} \frac{dx}{(x^2+1)^3} = 3\pi/16.$

(C) $\int_{-\infty}^{\infty} F(x)\sin mx\,dx$ *or* $\int_{-\infty}^{\infty} F(x)\cos mx\,dx$, *where F(x) is a rational function of x and m>0. Here, we consider* $\oint_C F(z)\,e^{imz}\,dz$, *where C is the same contour as that in Type (B).*

Here $\oint_C F(z)e^{imz}dz$ *can be used to evaluate* $\int_{-\infty}^{\infty} F(x)\sin mx\,dx$ *or* $\int_{-\infty}^{\infty} F(x)\cos mx\,dx$

6.1.2. Special Theorems Used in Evaluating Integrals :

Theorem 6.1.2.1. *If* $\lim_{z\to a}(z-a)f(z) = A$ *and if C is the arc* $\theta_1 \le \theta \le \theta_2$ *of the circle* $|z-a| = r$, *then*

$$\lim_{r\to 0}\int_C f(z)\,dz = i\,A(\theta_2-\theta_1)$$

In particular, if z = a is a simple pole of f(z), then A is the residue of f(z) at z = a and if C is the arc is a small circle $|z-a|=r$, *we have* $\theta_2 - \theta_1 = 2\pi$, *then we get* $\int_C f(z)\,dz = 2\pi i A$. *Particularly,*

if (z-a) f(z) $\to 0$ *as* $z \to 0$, *then* $\int_C f(z)\,dz \to 0$ *as* $z \to 0$. *In evaluating integrals as above, it is often necessary to show that* $\int_\Gamma f(z)\,dz$ *and* $\int_\Gamma e^{imz} f(z)\,dz$ *approaches to zero as* $R \to \infty$. *The following theorems are useful for it.*

Theorem 6.1.2.2. *If* $|F(z)| \le \frac{M}{R^k}$ *for* $z = Re^{i\theta}$, *where* $k > 1$ *and* M *is constant, then if* Γ *is the semicircle,*

$$\lim_{R\to+\infty}\int_{\Gamma} F(z)\,dz = 0$$

Theorem 6.1.2.3. *If* $|F(z)| \le \frac{M}{R^k}$ *for* $z = Re^{i\theta}$, *where* $k>0$ *and* M *is constant, then if* Γ *is the semicircle,*

$$\lim_{R\to+\infty}\int_{\Gamma} e^{imz}F(z)\,dz = 0$$

Some Important Integrals:

1. $$\int_{-\infty}^{\infty}\frac{\cos mx}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}, a>0, m\ge 0$$

2. $$\int_{-\infty}^{\infty}\frac{x\sin mx}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}(ma^2-1)e^{-ma}, a>0, m\ge 0$$

3. $$\int_{-\infty}^{\infty}\frac{\sin mx}{x(a^2+x^2)^2}dx = \frac{\pi}{2a^3}(1-e^{-|m|a}), a>0, m\in\mathbb{R}$$

4. $$\int_{0}^{\infty}\frac{\cos mx}{(a^2+x^2)}dx = \frac{\pi}{2a}e^{-|m|a}, a>0,\ m\in\mathbb{R}$$

5. $$\int_{-\infty}^{\infty}\frac{\cos mx}{(x^2+a^2)}dx = \frac{\pi}{a}e^{-|m|a}, a>0,\ m\in\mathbb{R}$$

6. $$\int_{-\infty}^{\infty}\frac{x\sin mx}{(x^2+a^2)}dx = s(m)\pi e^{-|m|a}, a>0,\ m\in\mathbb{R}$$, *where* $s(m)=1,-1,0$ *if* $m>0, m=0, m<0$ *respectively*

7. $$\int_{-\infty}^{\infty}\frac{\sin mx}{x(a^2+x^2)}dx = \frac{\pi}{2a^2}(1-e^{-m|a|}), m>0,\ a\in\mathbb{R}$$

6.1.3. *Jordan's Lemma Theorem:* *If* C_R *is semicircle with its centre at origin and radius R in the upper half-plane and f(z) satisfies the following conditions:*

(i) *It is analytic in the upper half-plane except at a finite number of poles.*

(ii) $f(z) \to 0$ *uniformly as* $|z| \to \infty$ *for* $0 \le \arg z \le \pi$, *then* $\lim_{R\to 0}\int_{C_R} e^{imz}f(z)\,dz = 0$, *where* m *is positive real number.*

6.1.4. Evaluation of Infinite Integrals when Integrand has Poles on Real Axis:

Let us consider the function $f(z)=\frac{e^{imz}}{z}$. Clearly the function f(z) has simple pole at z=0 and no pole in the upper half plane. The semicircular C, is chosen as shown in figure given below consisting of

(i) the semicircle C_R with centre at the origin and radius R in the upper half – plane given by $|z|=R$.

(ii) the line segment of the real axis from $-R$ to $-r$.

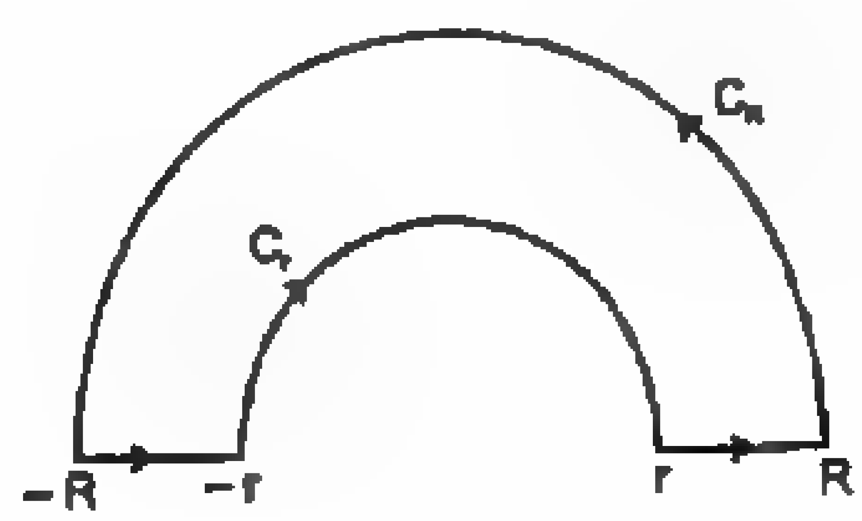
(iii) the semicircle C_r, with centre at the pole z=0 and of the infinitesimal radius r given by $|z|=r$.

(iv) the line segment of the real axis from r to R.

As there is no singularity in the interior of C, by Cauchy residue theorem

$$\int_C f(z)dz=\int_{C_R} f(z)dz+\int_{-R}^{-r} f(x)dx+\int_{C_r} f(z)dz+\int_r^R f(x)dx=0 \quad ...(1)$$

$$As\ |f(z)|=\left|\frac{e^{imz}}{z}\right|=\frac{1}{|z|}=\frac{1}{R}\to 0\ as\ R\to\infty;$$



Therefore by Jordan's lemma $\lim\limits_{R\to\infty}\int_{C_R} f(z)dz=\lim\limits_{R\to\infty}\int_{C_R}\frac{e^{imz}}{z}dz \quad ...(2)$

Now, on semicircle C, $z=re^{i\theta}\Rightarrow\frac{dz}{z}=id\theta$ and hence

$$\lim_{r\to0}\int_{C_R} f(z)dz=\lim_{r\to0}\int_{\pi}^{0} e^{imr(\cos\theta+i\sin\theta)}id\theta=\lim_{r\to0}\int_{\pi}^{0} e^{-mr\sin\theta}e^{imr\cos\theta}id\theta=-i\pi \quad ...(3)$$

Let $R\to\infty$ and $r\to0$ in equation (1), in view of (2) and (3)

$$\int_C f(z)dz=0+\int_{-\infty}^{0} f(x)dx-i\pi+\int_0^{\infty} f(x)dx=0\quad i.e.\ \int_{-\infty}^{+\infty} f(x)dx=i\pi\ \ or\ \int_{-\infty}^{+\infty}\frac{e^{imx}}{x}=i\pi$$

6.2. CONFORMAL MAPPING

Let $w=f(z)=u+iv$ be a complex valued function of a complex variable z, where $z=x+iy$ and $u=u(x,y)$, $v=v(x,y)$, then some correspondence between point (u,v) in the w-plane with the point (x,y) in the z-plane is obtained which is called a **mapping** or transformation of points in z-plane into points of the w-plane.

Note:

(a) *If for each point of z-plane, there corresponds one and only one point of w-plane, then such correspondence is known as one to one transformation.*

(b) *The corresponding sets of points in the two planes are called images of each other.*

(c) $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \dfrac{\partial u \partial v}{\partial x \partial y} - \dfrac{\partial u \partial v}{\partial y \partial x}$, *is called the Jacobian of the transformation.*

If u and v are continuously differentiable in any region D and if the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$ *does not vanish in D, then the transformation is one to one. In particular, if f(z) is an analytic function, then using Cauchy-Riemann equations* $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, *we get* $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial u}{\partial x} \cdot \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right)$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right|^2 = |f'(z)|^2$$

6.2.1. *CONFORMAL AND ISOGONAL TRANSFORMATION:*

Let the transformation u=u(x,y), v=v(x,y) map a point $z_0=x_0+iy_0$ *of the z plane to a point* $w_0=u_0+iv_0$ *of the w plane. Further let the two curves* C_1 *and* C_2 *intersecting at* z_0 *be mapped on two curves* γ_1 *and* γ_2 *intersecting at* w_0*. If the angle between the intersecting curves* C_1 *and* C_2 *at* z_0 *is equal to the angle between the intersecting curves* γ_1 *and* γ_2 *at* w_0*, then the transformation is called isogonal transformation.*

However if the sense of rotation as well as the magnitude of the angle is preserved, then the transformation is said to be conformal.

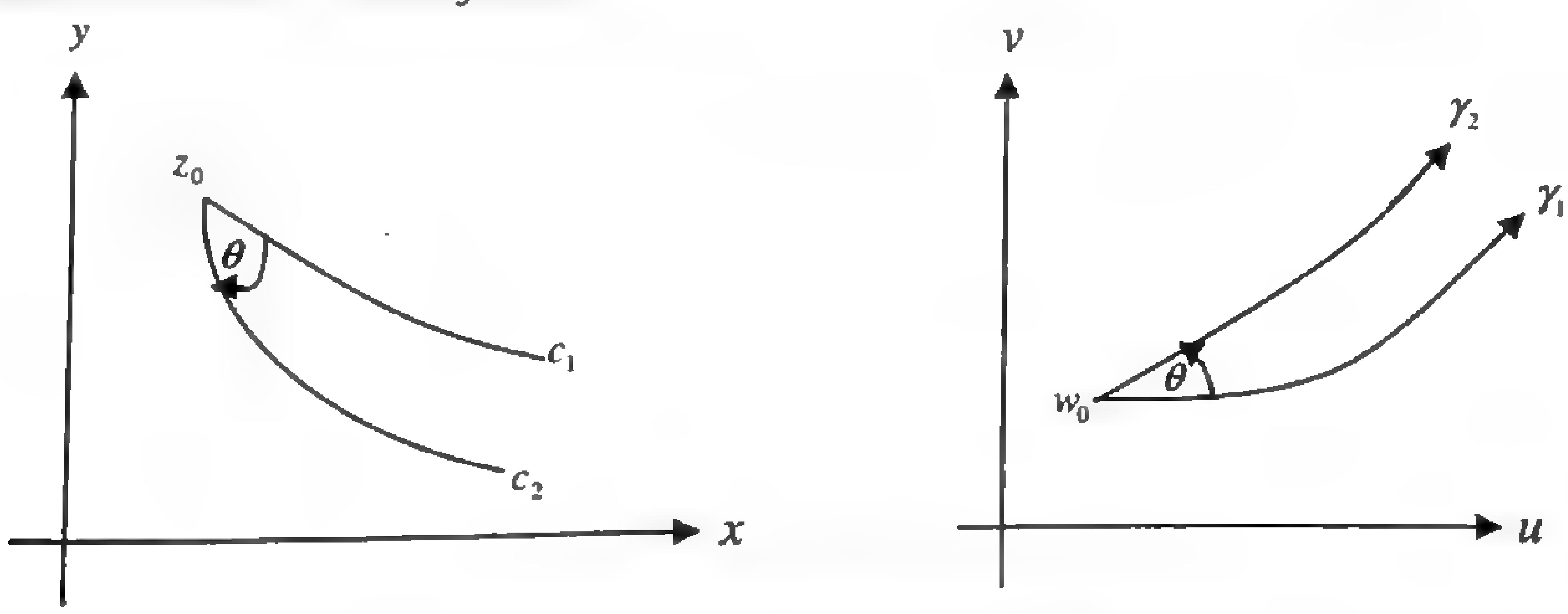


Fig: Isogonal Transformation

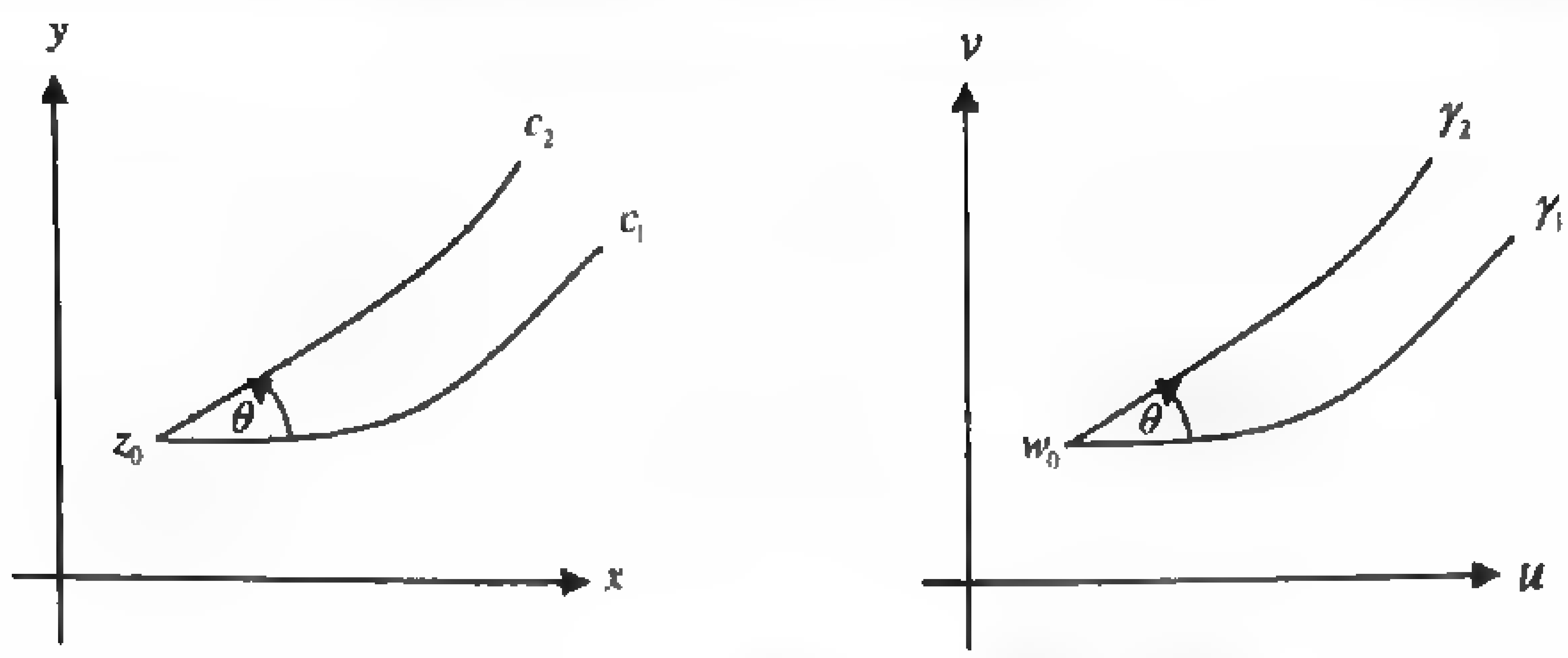


Fig: Conformal Transformation

Necessary and Sufficient Conditions for w= f(z) to Represent a Conformal Mapping:

(a) *If w=f(z) represents a conformal transformation of a domain D in the z-plane into a domain D' of the w-plane, then f(z) is an analytic function of z in D, i.e., if f(z) = u(x,y) + i v (x,y) is conformal, then f(z) must be an analytic function of z. Thus u(x,y) and v(x,y) are differentiable functions and hence must satisfy Cauchy – Riemann equations, i.e.,* $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

(b) *Let f(z) be an analytic function of z in a region D of the z-plane and let $f'(z)\neq 0$ inside D. Then, the mapping w = f(z) is conformal at the points of D.*

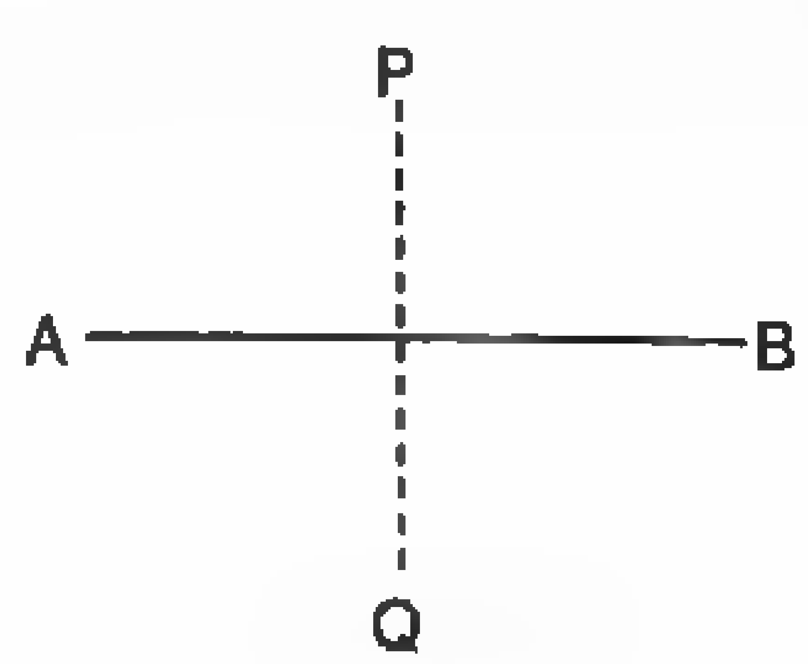
Note:

(1) *If f(z) is analytic at z_0 and $f'(z_0)\neq 0$, then under the transformation w=f(z), the tangent at z_0 to any curve C_l is rotated through an angle $\theta_0 = \arg\{f'(z_0)\}$. Also, the distance of a point z_l on C_l from the point z_0 is magnified by an amount $|f'(z_0)|$.*

(2) *If f(z)= w conformal at z_0, then f(z) has local inverse at z_0*

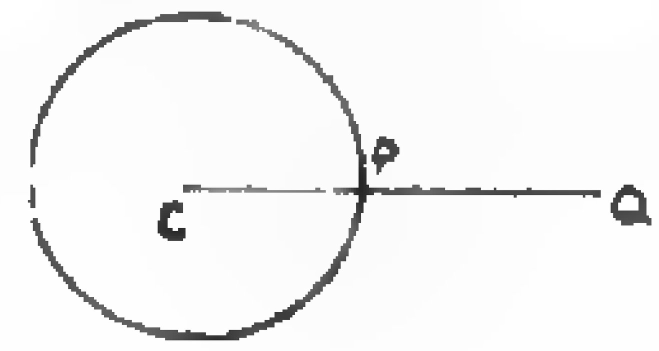
6.3. INVERSE POINTS

6.3.1. Inverse points with respect to a line: *The two points P and Q are said to the inverse points with respect to a line AB if Q is the image of P in AB, i.e., if the line AB is the right angle bisector of PQ.*



6.3.2. *Inverse points with respect to a circle:*

Two points P and Q are said to be the inverse points with respect to a circle S if they are collinear with the centre C and on the same side of it and if the product of their distances from the centre is equal to r^2, where r is the radius of the circle. Thus when P and Q are the inverse points of the circle, then the three points C, P, Q are collinear, and also $CP.CQ = r^2$, where P and Q lie on the same side of C.



Remarks:

1. *Two points z_1, z_2 are the inverse points with respect to the circle $z\bar{z} + \bar{b}z + b\bar{z} + c = 0$ iff $z_1\bar{z}_2 + b\bar{z}_2 + \bar{b}z_1 + c = 0$*

2. *Inverse of a point 'a' with respect to the circle $|z - c| = R$ (circle of radius R with centre at the point c). is the point $c + \dfrac{R^2}{\bar{a} - \bar{c}}$*

3. *If centre of the circle is at the origin and radius is R, then $b = \dfrac{R^2}{\bar{a}}$*

4. *If centre is at the origin and radius unity, then $b = \dfrac{1}{\bar{a}}$.*

6.4. *BILINEAR TRANSFORMATION*

6.4.1. *Some General Transformations:* *In the following α, β are given complex constants while a, θ_0 are real constants.*

1. ***Translation:*** $w = z + \beta$. *This transformation will displace or translate every point in the z plane along the direction of β through a distance equal to |β|.*

2. ***Rotation:*** $w = \beta z, \beta \in \mathbb{C}$
By this transformation, the curves in the plane are rotated through an equal angle to arg β, where β is unimodular, i.e., w=βz is rotation of |β|=1.

3. ***Stretching or Magnification:*** $w = az, a \in \mathbb{R}$
By this transformation, curves in the z-plane are stretched (or contracted) in the direction of z in ratio |a| if a > 1 (or 0 < a < 1). Contraction as a special case of stretching. This transformation is also known as homothetic transformation. w=az, a∈ℂ represents magnification if arg a=zero, i.e., if a is real and positive.

4. ***Inversion:*** *$w = 1/z$. Clearly $w=(1/z)$ is the resultant of $z=1/z, w=1/z$, where the formal is reflection in the real axis and the latter is reflection in unit circle.*

6.4.2. *Linear Transformation:* *The transformation $w = \alpha z + \beta$ ...(1), where α and β are given complex constants, is called a linear transformation. Thus a general linear transformation is a combination of the translation, rotation and magnification.*

6.4.3. *Bilinear or Fractional or Mobius Transformation:* *The transformation $w=\dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ ...(2) is called a bilinear or fractional transformation. This transformation associates a unique point of the w-plane to each point of z-plane and conversely. This transformation can be considered as the resultant of a series of translations, rotation, magnification and inversions.*

Bilinear transformation (2) has the property that circles in the z-plane are mapped into circles in the w-plane, where by circle we can include circle of infinite radius which are straight lines.

Note: *1) A given bilinear transformation is resultant of an even number of inversions.*
2) The expression ad-bc is called the determinant of bilinear transformation (2).

Result: Product of Two Bilinear Transformations is Bilinear Transformation:

Consider two bilinear transformations T_1 and T_2 defined by

$$T_1(z)=\frac{a_1z+b_1}{c_1z+d_1},\ (a_1d_1-b_1c_1\neq 0) \text{ and } T_2(z)=\frac{a_2z+b_2}{c_2z+d_2},\ (a_2d_2-b_2c_2\neq 0)\text{, then}$$

$$T_2(T_1(z))=T_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right)=\frac{a_2\left(\dfrac{a_1z+b_1}{c_1z+d_1}\right)+b_2}{c_2\left(\dfrac{a_1z+b_1}{c_1z+d_1}\right)+d_2}$$

Hence, $e=T_2(T_1(z))=\dfrac{\alpha z+\beta}{\gamma z+\delta}$, *where* $\alpha=a_2a_1+b_2c_1, \beta=a_2b_1+b_2d_1,$

$\gamma=c_2a_1+d_2c_1, \delta=c_2b_1+d_2d_1$

The determinant is $\alpha\delta-\beta\gamma$

$=(a_2a_1+b_2c_1)(c_2b_1+d_2d_1)-(a_2b_1+b_2d_1)(c_2a_1+d_2c_1)$

$=a_2a_1d_2d_1+b_2b_1c_2c_1-a_2b_1d_2c_1-b_2d_1c_2a_1$

$=(a_1d_1-b_1c_1)(a_2d_2-b_2c_2)$

$\neq 0$

It follows that the transformation T_2T_1 is also a billinear transformation and is called product of two transformations T_1 and T_2.

6.4.4. *Critical Points:* *Let f(z) be a non-constant analytic function in a domain D. If $f'(z_0)=0$ for some z_0 in D, then z_0 is called the critical point of the transformation. Also, the points where $f'(z_0)\neq 0$ are called ordinary points*

Consider the bilinear transformation $w = T(z) = \dfrac{az+b}{cz+d}$ ...(1)

Solving this for z, we get the inverse map as $z = T^{-1}(w) = \dfrac{b-wd}{wc-a}$ ...(2)

Transformation T associates a unique point of the w–plane to any point in z- plane except the point $z = -\frac{d}{c}$ *when* $c \neq 0$. *The transformation* T^{-1} *associates a unique point of z- plane to any point of w-plane except the point* $w = \frac{a}{c}$, *when* $c \neq 0$. *These exceptional points* $z = -\frac{d}{c}$ *and* $w = \frac{a}{c}$ *are mapped into the points* $w = \infty$ *and* $z = \infty$ *respectively as obvious from (1) and (2)*

From (1), $\dfrac{dw}{dz} = \dfrac{ad-bc}{(cz+d)^2} \Rightarrow \dfrac{dw}{dz} = \begin{cases} \infty & if \quad z = -\dfrac{d}{c} \\ 0 & if \quad z = \infty \end{cases}$

The points $z = -\dfrac{d}{c}$, $z = \infty$ *are called critical points, where the conformal property does not hold good.*

If the complex plane is closed by addition of the point ∞, *then we say that the bilinear transformation sets up one-one correspondence between all the points of the closed z-plane and closed w-plane.*

6.4.5. Fixed or Invariant Points of a Transformation: *The points which coincide with their transformation are called invariant or fixed points of the transformation. Thus, fixed points of a transformation* $w = f(z)$ *are obtained by putting* $w = z$

The invariant points of the transformation $w = \dfrac{az+b}{cz+d}$ ...(1)

is given by $z = \dfrac{az+b}{cz+d}$ *or* $cz^2 - (a-d)z - b = 0$ *or* $z = \dfrac{(a-d)\pm\sqrt{M}}{2c}$ ...(2)

where $M = (a-d)^2 + 4bc$. *The number of finite fixed points is one or two according as* $M = 0$ *or* $M \neq 0$.

Case I: *When* $c \neq 0$

(i) *If* $M \neq 0$, *then (1) have two fixed points given by* $z = \dfrac{(a-d)\pm\sqrt{M}}{2c}$

(ii) *If* $M = 0$, *then (1) has only one finite fixed point given by* $z = \dfrac{(a-d)}{2c}$

Case II: *When* $c = 0$

(i) *If* $d \neq 0$, *then (1) becomes* $w = \dfrac{az+b}{0+d} = \dfrac{a}{d}z + \dfrac{b}{d}$

The fixed point is given by $z = \dfrac{a}{d}z + \dfrac{b}{d}$ *or* $z = \dfrac{b}{d-a}$ ...(3)

(ii) *If* $a-d \neq 0$, *then (2) declares that one fixed point is* ∞ *and (3) declares that the other fixed point is finite.*

(iii) *If* $a-d = 0$, *then the transformation has one fixed point, i.e,* ∞, *according to (3) .*

Thus we have the following results:

(i) *if $c \neq 0$ and $M \neq 0$, two finite fixed points.*

(ii) *if $c \neq 0$ and $M = 0$, one finite fixed point.*

(iii) *If $c = 0$ and $d \neq 0$, only one finite fixed point.*

(iv) *if $c= 0$ and $a\text{-}d = 0$, only one infinite fixed point ,i.e., ∞. In this case $w = az + (b/d)$.*

(v) *if $c=0$ and $a\text{-}d\neq 0$, one finite and the other is ∞.*

Normal Form of a Bilinear Transformation:

(i) *Every bilinear transformation with two finite fixed points α, β can be put in the form $\dfrac{w-\alpha}{w-\beta} = k\dfrac{z-\alpha}{z-\beta}$*

and if $\beta = \infty$, then it becomes $w - \alpha = k(z - \alpha)$

(a) *If $|k|=1$, then transformation is called elliptic.*

(b) *If $k > 0 (\neq 1)$, then transformation is called hyperbolic.*

(c) *If $k = a.e^{i\alpha}$, where a and α are real numbers such that $a \neq 1$, $a > 0$ and $\alpha \neq 0$, then transformation is called loxodromic.*

(ii) *Every bilinear transformation which has only one finite fixed point α can be put in the form $\dfrac{1}{w-\alpha} = \dfrac{1}{z-\alpha} + k$ and if $\alpha = \infty$, then it becomes $w = z + k$*
In this case, transformation is called parabolic.

Example 6.4.5.1. *Find the fixed points and the normal form of the following bilinear transformations:*

(i) $w = \dfrac{z}{z-2}$ (ii) $w = \dfrac{z-1}{z+1}$

Is any of these transformations hyperbolic, elliptic or parabolic?

Solution:

(i) *The fixed points are given bay $w = z$, i.e., $z = \dfrac{z}{z-2}$ or $z^2 - 2z - z=0$ or $z(z - 3) = 0$.*

$\Rightarrow z = 0, 3$. *Hence fixed points are 0, 3.*

To find normal form, $w-3=\dfrac{z}{z-2}-3= \dfrac{-2z+6}{z-2}$ or $\dfrac{w-0}{w-3} = \left(\dfrac{z}{z-2}\right)\left(\dfrac{z-2}{6-2z}\right) = -\dfrac{1}{2}\left(\dfrac{z-0}{z-3}\right)$

or $\dfrac{w-0}{w-3} = k\left(\dfrac{z-0}{z-3}\right)$ with $k = -\dfrac{1}{2} = \dfrac{1}{2}e^{i\pi}$, *hence the given transformation is loxodromic.*

(ii) *The fixed points are given by* $z = \dfrac{z-1}{z+1}$

$z^2 + z - z + 1 = 0 \Rightarrow (z-i)(z+i) = 0 \Rightarrow z=i$ or $z= -i$

$$w - i = \frac{z-1}{z+1} - i = \frac{(1-i)z-(1+i)}{z+1} \qquad ...(1)$$

and $$w + i = \frac{z-1}{z+1} + i = \frac{(1+i)z-(1-i)}{z+1} \qquad ...(2)$$

Dividing (1) by (2), we get $\frac{w-i}{w+i}=\frac{1-i}{1+i}\left[\frac{z-(1+i)/(1-i)}{z-(1-i)/(1+i)}\right]$ *or* $\frac{w-i}{w+i}=-i\frac{z-i}{z+i}$ *, which is in the normal form.*

$\left[\because \frac{1-i}{1+i}=\frac{(1-i)^2}{1+1}=\frac{1-1-2i}{2}=-i\right]$

Here $k=-i=e^{-i\pi/2} \Rightarrow |k|=1$, *so transformation is elliptic.*

6.4.6. Cross Ratio: *If* z_1, z_2, z_3, z_4 *are distinct point, then the ratio* $\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$ *is called the cross ratio of* z_1, z_2, z_3, z_4 *and is denoted by* (z_1, z_2, z_3, z_4)*. A bilinear transformation preserves cross ratio, i.e., if* z_1, z_2, z_3, z_4 *are transformed to* w_1, w_2, w_3, w_4 *respectively, then* $(z_1, z_2, z_3, z_4)=(w_1, w_2, w_3, w_4)$.

Equation of a Circle through three given points:

Equation of a circle passing through three given points z_1, z_2 *and* z_3 *is given by*

$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}=\frac{(\bar{z}-\bar{z}_1)(\bar{z}_2-\bar{z}_3)}{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-\bar{z})}$, *where z is any complex variable lying on circle*

6.4.7. *Some Important Theorems:*

Theorem 6.4.7.1. *Every bilinear transformation is the resultant of magnification, rotation, inversion and translation.*

Proof: *Let* $w=T(z)=\frac{az+b}{cz+d}(ad-bc\neq 0)$, *be a bilinear transformation*

If $c\neq 0$, *this transformation can be written as* $w=\frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)}+\frac{a}{c}$

Let us write $Z=z+\frac{d}{c}$ ...(i)

$\xi=\frac{1}{Z}$ ...(ii)

$\tau=\frac{bc-ad}{c^2}\xi$ ...(iii)

$\Rightarrow w=\tau+\frac{a}{c}$ ...(iv)

Above relations show that the bilinear transformation is the resultant of translation given by equation (i), inversions in the real axis and unit circle given by equation (ii), a rotation and magnification given by equation (iii) and then again translation given by equation (iv). If c=0, then $T(z)=\frac{a}{d}z+\frac{b}{d}$ *provided* $d\neq 0$

Let $\xi = \frac{a}{d}z$. *Then,* $w = \xi + \frac{b}{d}$

Thus, the given transformation is the resultant of magnification, rotation and translation.

Note: *Inverse of bilinear transformation is again bilinear transformation.*

Theorem: 6.4.7.2. *The four points* z_1, z_2, z_3 *and* z_4 *are concyclic, if* $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ *is purely real.*

Theorem: 6.4.7.3. *The set of all Bilinear Transformations forms a non-abelian group under the product of transformations.*

Theorem: 6.4.7.4. *Bilinear Transformation maps family of circles and straight lines to family of circles and straight lines.*

Example: 6.4.7.4.1. *The map* $w=1/z$ *transforms circles and lines into circles and lines.*

Solution: *Let a,b,c,d be real numbers such that* $b^2+c^2> 4ad$

Then the circle in z-plane is represented by $a(x^2+y^2)+bx+cy+d=0$... ...(1)

taking $w=u+iv=1/z$, *(1) is transformed to* $d(u^2+v^2)+bu-cv+a=0$... ...(2)

which is a circle or line depending on the nature of real constants.

Thus from(2), it is concluded that

(a) *A circle not passing through the origin* ($a \neq 0$, $d \neq 0$) *in the z-plane is mapped on a circle in the w-plane not passing through the origin.*
(b) *A circle passing through the origin* ($a \neq 0$, $d = 0$) *in the z-plane is mapped on a line in the w-plane not passing through the origin.*
(c) *A line not passing through the origin* ($a = 0$, $d \neq 0$) *in the z-plane is mapped on a circle in the w-plane passing through the origin.*
(d) *A line passing through the origin* ($a = 0$, $d = 0$) *in the z-plane is mapped on a line in the w-plane passing through the origin.*
(e) *maps* $|z|$ *onto* $|w| = 1$
(f) *maps* $|z| < 1$ *to* $|w| > 1$ *and* $|z| > 1$ *to* $|w| < 1$

Theorem: 6.4.7.5. *Bilinear Transformation maps inverse points w.r.t. a circle to the inverse points of transformed circle.*

Theorem: 6.4.7.6. *Every Bilinear Transformation is the resultant of an even number of inversions.*

Theorem: 6.4.7.7. *Every bilinear transformation maps the family of circles* $\arg\left(\frac{z - z_1}{z - z_2}\right) = \mu(\mu \neq 0)$ *through the fixed points* z_1 *and* z_2 *onto a similar family in the w-plane through two fixed points* w_1 *and* w_2 *which are the transformation of the points* z_1 *and* z_2 *respectively.*

Theorem: 6.4.7.8. (Bilinear Transformation Theorem) *Let f(z) be analytic at z_0 such that $f'(z_0)=0$. If $f'(z)$ has a zero of order k-1,k=1,2,..., at z_0. Then mapping $w = f(z)$ magnifies the angle at the vertex z_0 by the factor k.*

Theorem 6.4.7.9. (Riemann Mapping Theorem) *Let Ω be a simple connected domain in the complex plane (except the plane itself). Then $\exists$ a conformal map which maps Ω in a one to one manner onto the unit disc $|w|<1$.*

Remark: *To find a bilinear transformation which maps z_1, z_2, z_3 to w_1, w_2, w_3. Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation. Then simplify $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.*

Example 6.4.7.1. *Consider the transformations $w = T_1(z) = \dfrac{z+2}{z+3}$, $w = T_2(z) = \dfrac{z}{z+1}$.*

Find $T_1^{-1}(w)$, $T_2^{-1}(w)$, $T_2T_1(z)$, $T_1T_2(z)$ and $T_2^{-1}T_1(z)$.

Solution: *Solving* $w=\dfrac{z+2}{z+3}$, *we get* $z=\dfrac{2-3w}{w-1}$ $\therefore T_1^{-1}(w)=\dfrac{2-3w}{w-1}$

$$T_2T_1(z) = T_2\left(\frac{z+2}{z+3}\right) = \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3}+1} = \frac{z+2}{2z+5}$$

Again, $w=\dfrac{z}{z+1}$ *gives* $z=\dfrac{-w}{w-1}$ $\therefore T_2^{-1}(w) = -\dfrac{w}{w-1}$

$$\text{and } T_1T_2(z) = T_1\left(\frac{z}{z+1}\right) = \frac{\frac{z}{z+1}+2}{\frac{z}{z+1}+3} = \frac{3z+2}{4z+3}$$

$$\text{Also, } T_2^{-1}T_1(z) = T_2^{-1}\left(\frac{z+2}{z+3}\right) = -\frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3}-1} = z+2$$

Example 6.4.7.2. *Find the bilinear transformation mapping z = 2, 1, 0 to w = 1, 0, i respectively.*

Solution: *Let z be mapped to w. Then (w, 1, 0, i) = (z, 2, 1, 0) or* $\dfrac{(w-1)(0-i)}{(1-0)(i-w)} = \dfrac{(z-2)(1-0)}{(2-1)(0-z)}$

$\Rightarrow (w-1)(-i)(-z) = (z-2)(i-w) \Rightarrow iz(w-1) = (i-w)(z-2)$

$\Rightarrow w[z(1+i)-2] = 2iz-2i = 2i(z-1) \Rightarrow w = 2i\,\dfrac{(z-1)}{z(1+i)-2}$.

Example 6.4.7.3. *Find the bilinear transformation mapping $z=\infty$, i, 0 to $w = 0$, i, ∞ respectively.*

Solution: *The bilinear transformation mapping $(z, z'=\infty, i, 0)$ to $(w, 0, i, w'=\infty)$ is given by*

$$(w, 0, i, w') = (z, z', i, 0) \text{ or } \frac{w(i-w')}{-i(w'-w)} = \frac{(z-z')i}{(z'-i)(-z)} \text{ or } \frac{w\left(\frac{i}{w'}-1\right)}{-i\left(1-\frac{w}{w'}\right)} = \frac{i\left(\frac{z}{z'}-1\right)}{-z\left(1-\frac{i}{z'}\right)}$$

Put $z' = w' = \infty$. Then $\frac{w(-1)}{-i(1)} = \frac{i(-1)}{-z} \Rightarrow \frac{w}{i} = \frac{i}{z} \Rightarrow w = \frac{-1}{z}$

6.4.8. Some Special Bilinear Transformations:

Type 1: *Determine the totality of bilinear transformations which maps the real axis to itself.*

Solution: *Let x_1, x_2, x_3 be transformed under such a bilinear transformation to 0, 1, ∞ respectively (points on real axis).*

Then $(w, 0, 1, \infty) = (z, x_1, x_2, x_3)$, i.e.,

$$\frac{w(1-\infty)}{-1(\infty-w)} = \frac{(z-x_1)(x_2-x_3)}{(x_1-x_2)(x_3-z)} \text{ or } \frac{w(0-1)}{-1(1-0)} = \frac{(z-x_1)(x_2-x_3)}{(x_1-x_2)(x_3-z)} \text{ or } w = \frac{(x_2-x_3)z - x_1(x_2-x_3)}{-(x_1-x_2)z + x_3(x_1-x_2)}$$

The Jacobian of this bilinear transformation is $(x_2-x_3)(x_1-x_2)x_3 - x_1(x_1-x_2)(x_2-x_3)$

$= (x_2-x_3)(x_1x_3 - x_2x_3) - (x_1^2 - x_1x_2)(x_2-x_3) = -(x_1-x_2)(x_2-x_3)(x_3-x_1) \neq 0$ *as x_1, x_2, x_3 are distinct,*

$\therefore$ *$a = (x_2-x_3)$, $b = -x_1(x_2-x_3)$, $c = -(x_1-x_2)$, $d = x_3(x_1-x_2)$ should be real, $a \neq 0$, $c \neq 0$.*

Note: *If $a = 0 \Rightarrow x_2 - x_3 = 0 \Rightarrow x_2 = x_3$ but x_2 and x_3 are mapped to 1 and ∞ respectively.*

$\Rightarrow$ This is not a map [$\because$ same element has two images.]

Also, if $Im(z) > 0$ has to be mapped to $Im(w) > 0$. Then $Im(w) = \frac{w-\overline{w}}{2i}$

$$= \frac{1}{2i}\left[\frac{az+b}{cz+d} - \overline{\left(\frac{az+b}{cz+d}\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{ac z\overline{z} + ad z + bc\overline{z} + bd - ac z\overline{z} - ad\overline{z} - bcz - bd}{|cz+d|^2}\right]$$

$$= \frac{1}{2i}\frac{(ad-bc)(z-\overline{z})}{|cz+d|^2}$$

$\Rightarrow Im(z) > 0$ is mapped to $Im(w) > 0$ iff $ad - bc > 0$.

And $Im(z) > 0$ is mapped to $Im(w) < 0$ iff $ad - bc < 0$.

Type 2: Determine the totality of bilinear transformation mapping $|z| \le 1$ into $|w| \le 1$.

Solution: *Let* $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ *be the required bilinear transformation.*

Now $w = 0$ *and* $w = \infty$ *are inverse points w.r.t.* $|w| = 1$. *As inverse points are mapped to inverse points. Their pre images* $z = -b/a$ *and* $z = -d/c$ *are also inverse points w.r.t* $|z| = 1$. *If we take* $\alpha = -b/a$, *then* $-d/c = 1/\bar{\alpha}$.

$$\therefore\ w = \frac{a}{c}\,\frac{z+b/a}{z+d/c} = \frac{a}{c}\left(\frac{z-\alpha}{z-1/\bar{\alpha}}\right) \Rightarrow w = \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right) \qquad \ldots(1)$$

Now as each point on $|z|=1$ *is mapped on* $|w|=1$ $\therefore$ *for* $z=1$,

$$\therefore\ |w| = \left|\frac{a\bar{\alpha}}{c}\right| \left|\frac{1-\alpha}{\bar{\alpha}-1}\right| = 1 \Rightarrow \left|\frac{a\bar{\alpha}}{c}\right| = 1,$$

$$\therefore\ \frac{a\bar{\alpha}}{c} = e^{i\lambda} \text{ for some real } \lambda.$$

$$\therefore\ (1) \text{ becomes, } w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right) \qquad \ldots(2)$$

Also as $\alpha = 0$, $\therefore$ α *must be an interior point of* $|z|=1$. *Thus the totality of all such bilinear transformations is given by* $w = e^{i\lambda}\dfrac{z-\alpha}{\bar{\alpha}z-1}$, *where* $|\alpha| < 1$, λ *real.*

Remark:

(i) *Transformations given by (2) are those which map* $|z| = 1$ *onto* $|w| = 1$.

(ii) $|z| < 1$ *goes to* $|w| > 1$ *if* $|\alpha| > 1$ *and to* $|w| < 1$ *if* $|\alpha| < 1$.

(iii) *If* $z = 0$ *is mapped to* $w = 0$, *then* $0 = e^{i\lambda}\left(\dfrac{0-\alpha}{0-1}\right) = e^{i\lambda} z$

Hence $\alpha = 0$ *and the transformation is* $w = -e^{i\lambda} z$, *rotation.*

(iv) *If* $z - 1, -1$ *are mapped to* $w = 1, -1$ *respectively, then* $1 = e^{i\lambda}\left(\dfrac{1-\alpha}{\bar{\alpha}-1}\right)$ *and* $-1 = e^{i\lambda}\left(\dfrac{-1-\alpha}{-\bar{\alpha}-1}\right)$

so that on eliminating $e^{i\lambda}$, *we get* $1 = -\dfrac{\bar{\alpha}+1}{1+\alpha}\cdot\dfrac{1-\alpha}{\bar{\alpha}-1}$

$\Rightarrow \alpha\bar{\alpha} - \alpha + \bar{\alpha} - 1 = -\bar{\alpha} + \bar{\alpha}\alpha - 1 + \alpha \ \Rightarrow \alpha = \bar{\alpha} \Rightarrow \alpha$ *is real.*

and $w = e^{i\lambda}\dfrac{z-\alpha}{\bar{\alpha}z-1}$ *and* $z = 1$ *is mapped to* $w = 1$ *gives* $1 = e^{i\lambda}\dfrac{1-\alpha}{\alpha-1}$ *or* $e^{i\lambda} = -1$.

$$\therefore\ w = \frac{z-\alpha}{\bar{\alpha}z-1} = \frac{\alpha-z}{\bar{\alpha}z-1}$$

Type 3: Find all bilinear transformations which maps $Im(z) \geq 0$ onto $|w| \leq 1$.

Solution: Let $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ ...(1)

be the required transformation. Then $c \neq 0$ because if $c = 0$, then $z = \infty$ is mapped to $w = \infty$. But $z = \infty$ lies in $Im(z) \geq 0$ and $w = \infty$ does not lie in $|w| \leq 1$. So, $c \neq 0$. Also $w = 0$, $w = \infty$ are inverse points w.r.t $|w| = 1$.

$\therefore$ $z = -b/a$ and $z' = -d/c$ are inverse points w.r.t $Im(z)=0$, the real axis.

$\therefore -\dfrac{d}{c} = \overline{\left(\dfrac{-b}{a}\right)}$, if $\alpha = \dfrac{-b}{a}$, then $\overline{\alpha} = -\dfrac{d}{c}$

$\therefore$ (1) can be written as $w = \dfrac{a}{c}\left(\dfrac{z-\alpha}{z-\overline{\alpha}}\right)$ ... (2)

As $Im(z) = 0$ corresponds to $|w| = 1$, the point $z = 0$ corresponds to a point on $|w| = 1$.

$\therefore |w| = \left|\dfrac{a}{c}\right|\left|\dfrac{z-\alpha}{z-\overline{\alpha}}\right| = \left|\dfrac{a}{c}\right| = 1.$

$\therefore a/c = e^{i\lambda}$, λ real

$\therefore w = e^{i\lambda}\left(\dfrac{z-\alpha}{z-\overline{\alpha}}\right)$, λ real.

Also, $z = \alpha$ is mapped to $w = 0$, which lies inside $|w| = 1$. Hence $Im(\alpha) > 0$.

$w = e^{i\lambda}\left(\dfrac{z-\alpha}{z-\overline{\alpha}}\right)$, λ real, $Im(\alpha) > 0$.

Example 6.4.8.1. *Find the image of the circle $|z-3| = 3$ under the Mobius transformation $w = \dfrac{z}{z+1}$*

Solution: $w = \dfrac{z}{z+1}$ gives $wz + w = z$

$\Rightarrow z(w-1) = -w \Rightarrow z = \dfrac{w}{1-w}$ ...(1)

Now $|z-3| = 3$ gives $(z-3)\left(\overline{z}-3\right) = 3^2$ ...(2)

By (1), $z - 3 = \dfrac{w}{1-w} - 3 = \dfrac{4w-3}{1-w}$

$\overline{z} - 3 = \dfrac{\overline{w}}{1-\overline{w}} - 3 = \dfrac{4\overline{w}-3}{1-\overline{w}}$

Putting value in (2), $\dfrac{(4w-3)(4\overline{w}-3)}{(1-w)(1-\overline{w})} = 9 \Rightarrow 16w\overline{w} - 12w - 12\overline{w} + 9 = 9[1 + w\overline{w} - (w+\overline{w})]$

$\Rightarrow 7w\overline{w} - 3(w+\overline{w}) = 0 \Rightarrow 7(u^2+v^2) - 3(2u) = 0 \Rightarrow u^2 + v^2 - \dfrac{6}{7}u = 0$

Circle with center (-g,-f) = $\left(\frac{6}{7},0\right)$ *and radius* = $(g^2+f^2-c)^{\frac{1}{2}} = \frac{6}{7}$

Thus image is a circle with center $\left(\frac{6}{7},0\right)$ *and radius* = $\frac{6}{7}$ *in w – plane.*

Example 6.4.8.2. *Find the bilinear transformation which maps z=1, i, -1 respectively onto w=i, 0, -i. Also find image of |z|<1*

Solution: *Let z is mapped to w. Then (w, i, 0, -1) = (z, 1, i, -1)*

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \frac{-w+i}{w+i} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow i\,(iz-z+i-1) - w\,(iz-z+i-1) = w\,(iz+z-i-1) + i\,(iz+z-i-1)$$

$$\Rightarrow -2iwz + 2w = 2iz + 2 \quad \Rightarrow w = \frac{1+iz}{1-iz}$$

This is the required bilinear transformation

Now we have to find the image of |z| < 1,

As, $w = \frac{1+iz}{1-iz} \Rightarrow w - wiz - 1 - iz = 0$

$$\Rightarrow z = \frac{-1+w}{i(1+w)} \quad \Rightarrow z = \frac{i(1-w)}{1+w} \qquad \text{...(1)}$$

As $|z| < 1 \Rightarrow z\bar{z} < 1$ ...(2)

and $\bar{z} = \frac{-i(1-\bar{w})}{1+\bar{w}}$ ...(3)

Using (1), (2) and (3), we get

$$\Rightarrow \frac{i(1-w)}{(1+w)} \cdot \frac{-i(1-\bar{w})}{1+\bar{w}} < 1 \Rightarrow 0 < 2(w+\bar{w}) \Rightarrow 0 < (w+\bar{w})$$

$$\Rightarrow 0 < u \Rightarrow \operatorname{Re} w > 0$$

Thus |z| < 1 mapped to Rew> 0.

PRACTICE SET

Exercise 1. *The bilinear transformation w, which maps the points 0, 1, ∞ in the z-plane onto the points –i, ∞, 1, in the w plane, is*

(a) $\frac{z-1}{z+i}$ *(b)* $\frac{z-i}{z+1}$ *(c)* $\frac{z+i}{z-1}$ *(d)* $\frac{z+1}{z-i}$

Exercise 2. *Let w = f(z) be the bilinear transformation that maps -1, 0 and 1 to –i, 1 and -1 respectively, then f(1 – i) equals*

(a) $1 + 2i$ (b) $2i$ (c) $2 + i$ (d) $1 + i$

Exercise 3. Let $f(z)=\dfrac{2z+3}{z+3}, z \in \mathbb{C}$, then

(a) f maps H^+ to H^+, H^- to H^-
(b) f maps H^+ to H^-, H^- to H^+
(c) f maps L^+ to L^+, L^- to L^-
(d) f maps L^+ to L^-, L^- to L^+

Exercise 4. The function $f(z) = z^2$ maps the first quadrant onto

(a) itself
(b) upper half plane
(c) third quadrant
(d) right half plane

Exercise 5. Critical points of $f(z)=\dfrac{1+z^2}{1-z^2}$ is/are

(a) ± 1 (b) 0 (c) only ± 1 (d) $\pm 1, 0$

KEY POINTS

- *w=f(z)=u+iv is said to be a conformal mapping or transformation of a domain D in z-plane into a domain D' of w-plane if it preserves angle between two points in D both in terms of direction and magnitude.*

- *The mapping w=f(z) is conformal in a domain D if* $f'(z) \neq 0$ *inside D.*

- *The points at which the map is not conformal are called critical points.*

- $w=\dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$. *is said to be bilinear transformation if* $ad - bc \neq 0$. *It is a combination of translation, rotation, magnification and inversion mappings.*

- *Two points P and Q are said to be inverse points w.r.t a line AB if line AB is right bisector of PQ.*

- *Two points P and Q are inverse points w.r.t a circle with centre C and radius r if*
  (a) *they are collinear with center C.*
  (b) *They are on same side of C.*
  (c) $|CP|\,|CQ| = r^2$

- *Bilinear transformation preserves cross ratio.*

- *Bilinear transformation maps a family of circles and straight lines to a family of circles and straight lines.*

- *Four points z_1, z_2, z_3, z_4 lie on a circle if their cross ratio is purely real.*

- *If $w=\dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$ and $ad-bc \neq 0$, then real axis is mapped to real axis.*
  (i) *If $ad-bc>0$, then upper half plane is mapped to upper half plane.*
  (ii) *If $ad-bc<0$, then upper half plane is mapped to lower half plane.*

- *If $w=f(z)=e^{ir}\left(\dfrac{z-\alpha}{\bar{\alpha}z-1}\right)$, where $r \in \mathbb{R}$, then $|z|=1$ is mapped to $|w|=1$, $|z|<1$ is mapped to $|w|>1$ if $|\alpha|>1$ and $|z|<1$ is mapped to $|w|<1$ if $|\alpha|<1$*

- *If $w=f(z)=e^{ir}\left(\dfrac{z-\alpha}{z-\bar{\alpha}}\right)$, where $r \in \mathbb{R}$, then f maps $Imz=0$ to $|w|=1$, $Im(z)>0$ is mapped to $|w|<1$ if $Im(\alpha)>0$ and $Im(z)$ is mapped to $|w|>1$ if $Im(\alpha)<0$.*

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. *Let G_1 and G_2 be the images of the disc $\{z \in \mathbb{C}: |z+1|<1\}$ under the transformations $w=\dfrac{(1-i)z+2}{(1+i)z+2}$ and $w=\dfrac{(1+i)z+2}{(1-i)z+2}$ respectively. Then* **(GATE-2007)**

(a) $G_1=\{w \in \mathbb{C}: \mathrm{Im}(w)<0\}$ and $G_2=\{w \in \mathbb{C}: \mathrm{Im}(w)<0\}$
(b) $G_1=\{w \in \mathbb{C}: \mathrm{Im}(w)>0\}$ and $G_2=\{w \in \mathbb{C}: \mathrm{Im}(w)<0\}$
(c) $G_1=\{w \in \mathbb{C}: |w|>2\}$ and $G_2=\{w \in \mathbb{C}: |w|<2\}$
(d) $G_1=\{w \in \mathbb{C}: |w|<2\}$ and $G_2=\{w \in \mathbb{C}: |w|>2\}$

Solution: (b) *Given maps are bilinear transformations and under such maps lines and circles are transformed to lines and circles. Circles through origin map to lines and the boundary of disc is circle through origin.*

Under $w=\dfrac{(1-i)z+2}{(1+i)z+2}$; -1 is mapped to $w=i$.

i.e., interior point is mapped to exterior point

$\therefore$ option (a) is incorrect and hence option (b) is correct

Example 2. *Let f be a bilinear transformation that maps -1 to 1, i to ∞ and $-i$ to 0. Then $f(1)$ is equal to* **(GATE-2008)**

(a) -2 (b) -1 (c) i (d) $-i$

Solution: (b) $z_1=-1,\ w_1=f(z_1)=1$

$z_2=i,\ w_2=\infty=f(z_2)$

$z_3 = -i,\ w_3 = 0 = f(z_3)$

Since the bilinear transformation preserves cross ratio

$$\therefore \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\lim_{w_2\to\infty} \frac{(w-1)(w_2-0)}{(1-w_2)(0-w)} = \frac{(z+1)(i+i)}{(-1-i)(-i-z)}$$

$$\lim_{w_2\to\infty} \frac{(w-1)}{-w}\left(\frac{w_2}{1-w_2}\right) = \frac{(z+1)2i}{(-1-i)(-i-z)}$$

$$\Rightarrow \frac{w-1}{w} = \frac{2i}{1+i}\left(\frac{z+1}{z+i}\right) \quad \Rightarrow w = \frac{(i+1)(z+i)}{(z-1)-i(z+1)} = f(z)$$

$$\therefore f(1) = -1$$

Example 3. *Under the transformation* $w = \sqrt{\dfrac{1-iz}{z-i}}$, *the region* $D = \{z \in \mathbb{C} : |z| < 1\}$ *is transformed to*

(GATE-2010)

(a) $\{z \in \mathbb{C} : 0 < \arg z < \pi\}$

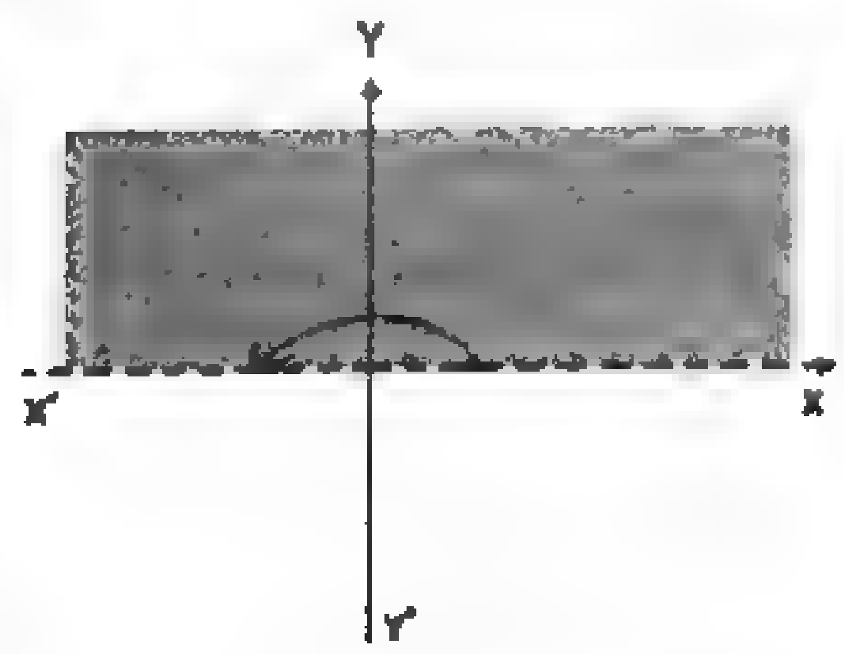
(b) $\{z \in \mathbb{C} : -\pi < \arg z < 0\}$

(c) $\left\{ z \in \mathbb{C} : 0 < \arg z < \dfrac{\pi}{2} \text{ or } \pi < \arg z < \dfrac{3\pi}{2} \right\}$

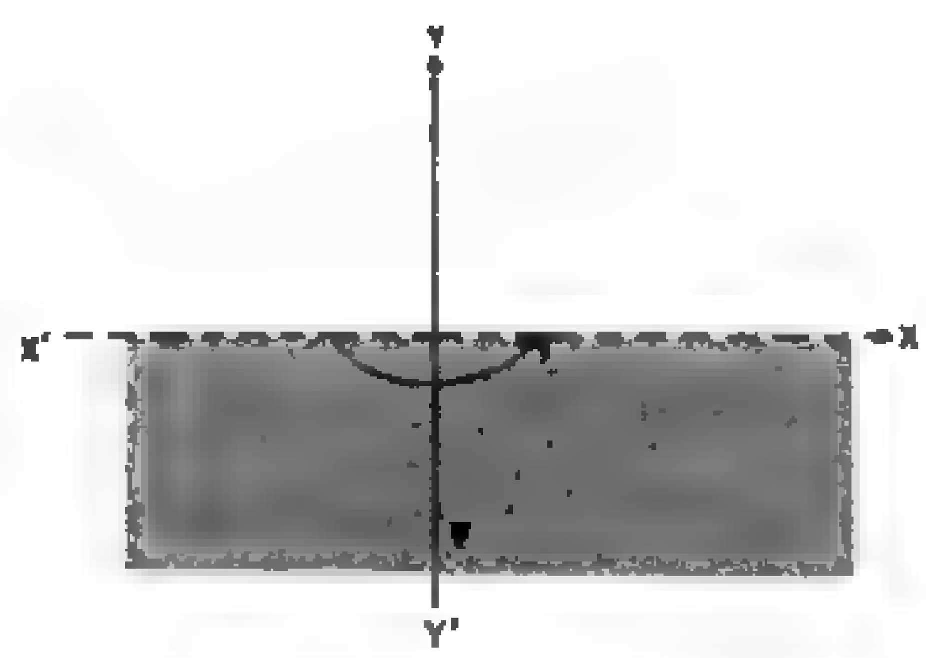
(d) $\left\{ z \in \mathbb{C} : \dfrac{\pi}{2} < \arg z < \pi \text{ or } \dfrac{3\pi}{2} < \arg z < 2\pi \right\}$

Solution: (c) $w = \sqrt{\dfrac{1-iz}{z-i}}$

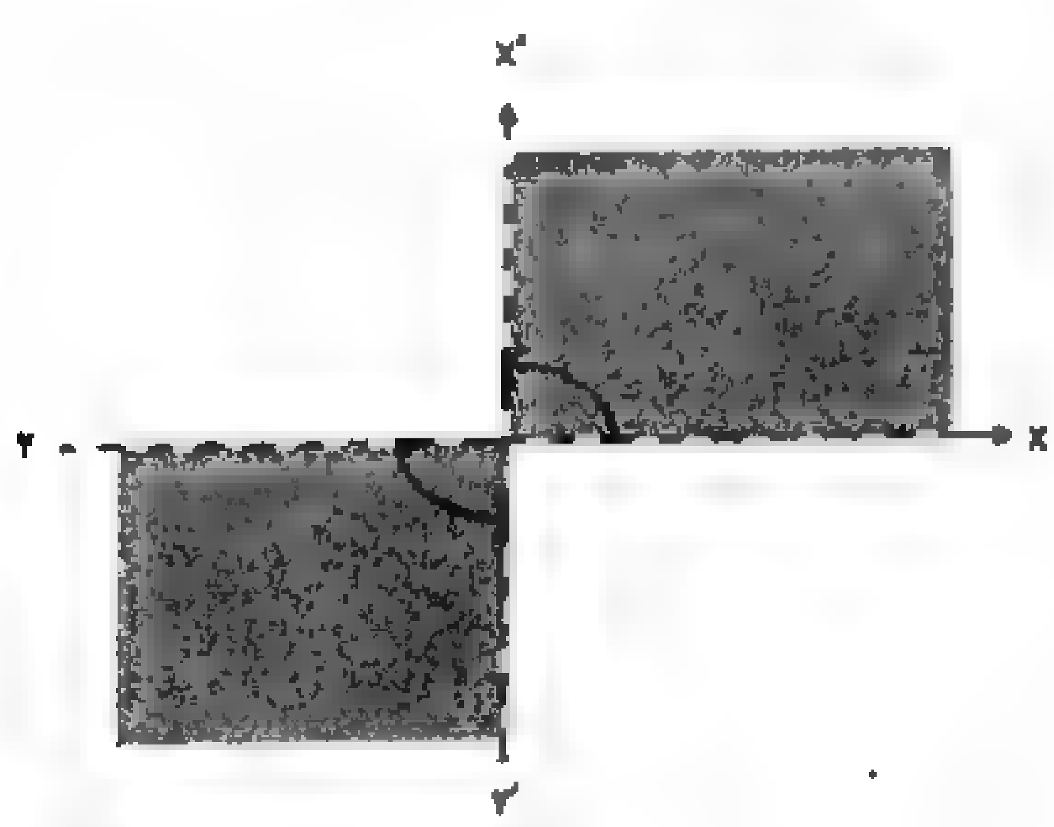
Region (1): $0 < \arg z < \pi$



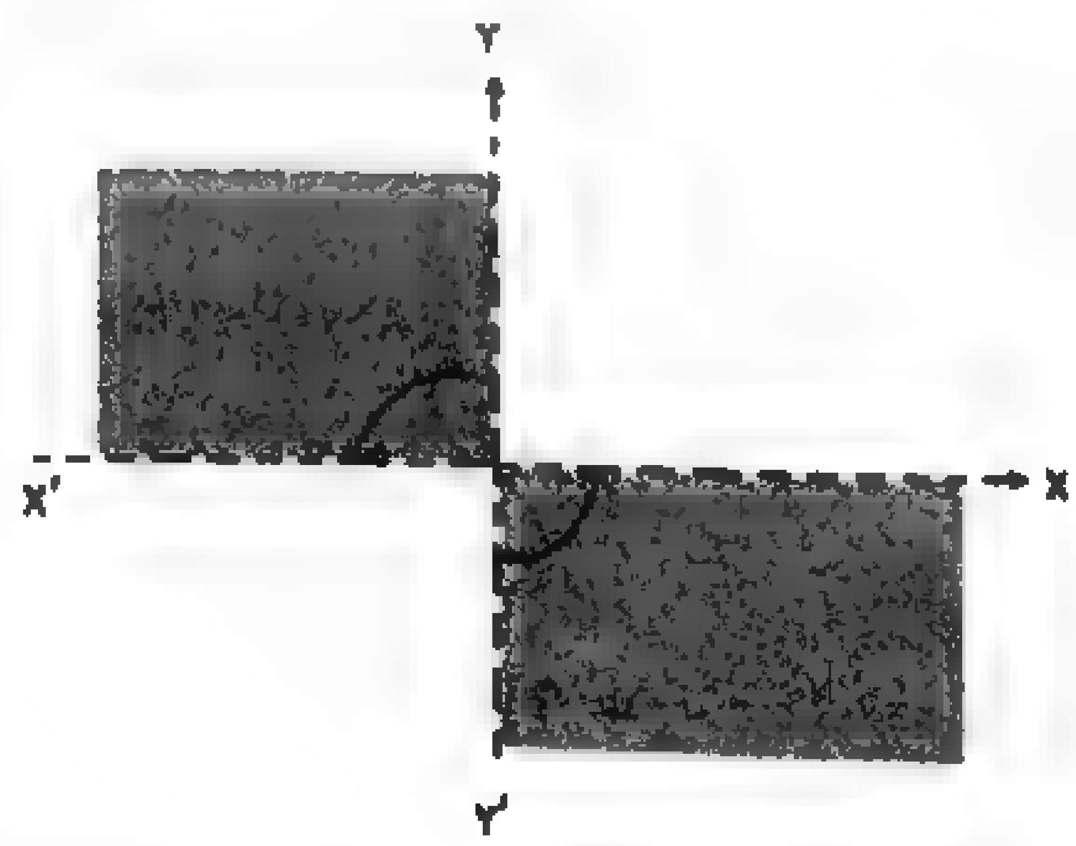
Region (2): $-\pi < \arg z < 0$



Region (3): $0 < \arg z < \dfrac{\pi}{2}$ *or* $\pi < \arg z < \dfrac{3\pi}{2}$



Region (4): $\dfrac{\pi}{2} < \arg z < \pi$ *or* $\dfrac{3\pi}{2} < \arg z < 2\pi$



Put $z=0$, $f(0)$ lies in first quadrant. So options (b) and (d) are incorrect.

[$\because z=0$ is interior point of D and under bilinear transformation interior point goes to interior point]

Put $z=-1 \Rightarrow f(-1)=i$. So option (a) is incorrect

$\Rightarrow$ Hence option (c) is correct

Example 4. *The straight lines $L_1: x=0$, $L_2: y=0$ and $L_3: x+y=1$ are mapped by the transformation $w=z^2$ into the curves C_1, C_2 and C_3 respectively. The angle of intersection between the curves at $w=0$ is*

(GATE - 2012)

(a) 0 (b) $\pi/4$ (c) $\pi/2$ (d) π

Solution: *(d) As we know that transformation $w=z^k$ magnifies the angle k times. Since the angle of intersection between L_1, L_2 and L_3 is $\frac{\pi}{2}$. So transformation $w=z^2$ magnifies the angle two times. Therefore angle of intersection at $w=0$ is π.*

Example 5. *Let $f:\mathbb{C}\setminus\{3i\}\to\mathbb{C}$ be defined by $f(z)=\frac{z-i}{iz+3}$. Which of the following statements about f is FALSE?*

(GATE-2013)

(a) f is conformal on $\mathbb{C}\setminus\{3i\}$

(b) f maps circle in $\mathbb{C}\setminus\{3i\}$ onto circle in $\mathbb{C}$

(c) All the fixed points of f are in the region $\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$

(d) There is no straight line in $\mathbb{C}\setminus\{3i\}$ which is mapped onto a straight line in $\mathbb{C}$ by f

Solution: *(c) $f:\mathbb{C}\setminus\{3i\}\to\mathbb{C}$ be $f(z)=\frac{z-i}{iz+3}$*

f(z) is conformal when $f'(z)\neq 0$ and f(z) is analytic. So f(z) is conformal in $\mathbb{C}\setminus\{3i\}$, then f(z) maps circles in $\mathbb{C}\setminus\{3i\}$ onto circles in $\mathbb{C}$. Fixed points are given by $\frac{z-i}{iz+3}=z$

$$\Rightarrow iz^2+3z=z-i \quad \Rightarrow iz^2+2z+i=0 \quad \Rightarrow z^2-2iz+1=0$$

$$\Rightarrow z=\frac{2i\pm\sqrt{-4-4}}{2} \quad \Rightarrow z=i(1\pm\sqrt{2})$$

Example 6. *The image of the region $\{z\in\mathbb{C}:\mathrm{Re}(z)>\mathrm{Im}(z)>0\}$ under the mapping $z\to e^{z^2}$ is*

(GATE-2013)

(a) $\{w\in\mathbb{C}:\mathrm{Re}(w)>0,\mathrm{Im}(w)>0\}$

(b) $\{w\in\mathbb{C}:\mathrm{Re}(w)>0,\mathrm{Im}(w)>0,|w|>1\}$

(c) $\{w\in\mathbb{C}:|w|>1\}$

(d) $\{w\in\mathbb{C}:\mathrm{Im}(w)>0,|w|>1\}$

Solution: *(c) The given region is $D=\{z\in\mathbb{C}: Re(z)>Im(z)>0\}$ and the mapping is $f(z)=e^{z^2}$*

Take $z=4+\pi i$

$$f(z)=e^{(4+\pi i)^2}=e^{16-\pi^2+8\pi i}=e^{16-\pi^2} \quad [\because e^{8\pi i}=1]$$

Here, Im(w)=0
So, option (c) is correct.

Example 7. *Let a,b,c,d* $\in \mathbb{R}$ *be such that* $ad - bc > 0$. *Consider the Mobius transformation* $T_{a,b,c,d}(z) = \dfrac{az+b}{cz+d}$.

Define $H_+ = \{z \in \mathbb{C} : Im\,(z) > 0\}$, $H_- = \{z \in \mathbb{C} : Im\,(z) < 0\}$

$R_+ = \{z \in \mathbb{C} : Re\,(z) > 0\}$, $R_- = \{z \in \mathbb{C} : Re(z) < 0\}$. *Then,* $T_{a,b,c,d}$ *maps* ***(CSIR UGC NET DEC-2008)***

(a) H_+ to H_+ (b) H_+ to H_- (c) R_+ to R_+ (d) R_+ to R_-

Solution: (a) *Let* $a = 1$, $d = 1$, $b=c=0$ *so that* $ad - bc > 0 \Rightarrow T_{a,b,c,d}(z) = z$

$\Rightarrow$ *options (b) and (d) are incorrect,*

Now take $a = 0$, $d = 0$, $b = -1$, $c = 1$ *so that* $ad - bc > 0$ *is satisfied.*

$$\Rightarrow T_{a,b,c,d}(z) = \frac{-1}{z} \Rightarrow T_{a,b,c,d}(2) = \frac{-1}{2}$$

$\therefore$ *option (c) is incorrect.*

Hence, option (a) is correct.

Example 8. *Define* $H^+ = \{z \in \mathbb{C} : y > 0\}$

$H^- = \{z \in \mathbb{C} : y < 0\}$

$L^+ = \{z \in \mathbb{C} : x > 0\}$

$L^- = \{z \in \mathbb{C} : x < 0\}$

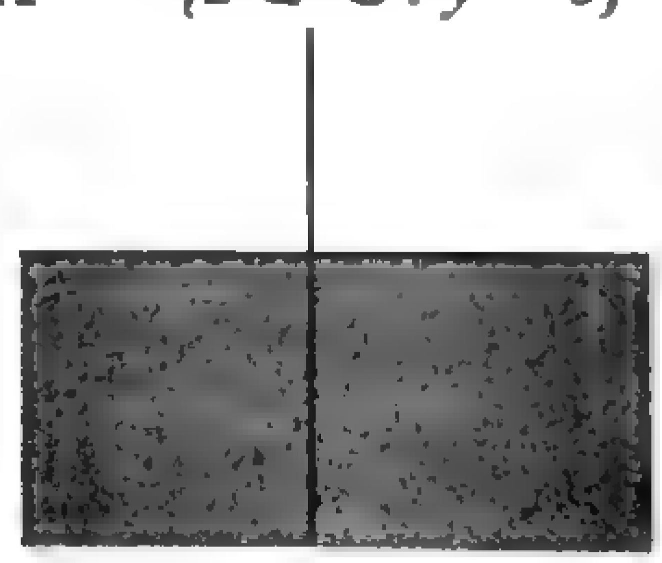
The function $f(z) = \dfrac{z}{3z+1}$ ***(CSIR UGC NET JUNE-2011)***

(a) maps H^+ onto H^+ and H^- onto H^- (b) maps H^+ onto H^- and H^- onto H^+

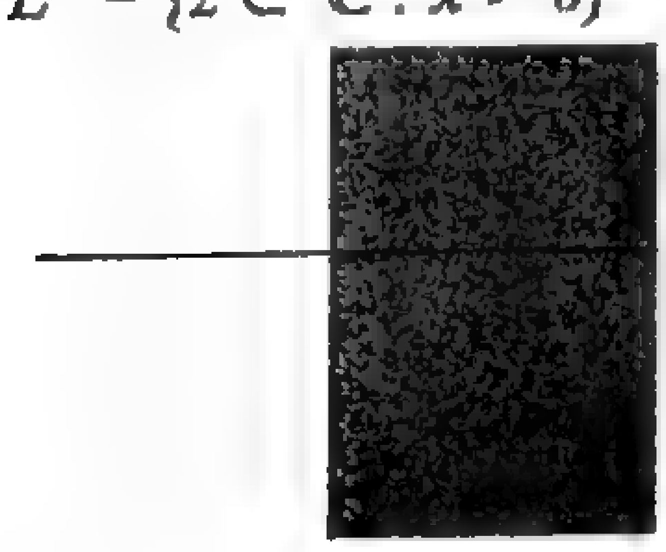
(c) maps H^+ onto L^+ and H^- onto L^- (d) maps H^+ onto L^- and H^- onto L^+

Solution: $H^+ = \{z \in \mathbb{C} : y > 0\}$

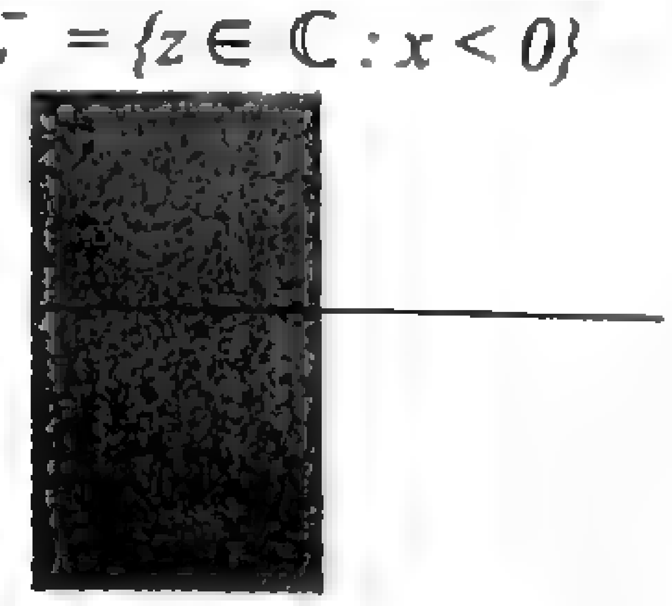
$H^- = \{z \in \mathbb{C} : y < 0\}$



$L^+ = \{z \in \mathbb{C} : x > 0\}$



$L^- = \{z \in \mathbb{C} : x < 0\}$



Given $f(z)=\dfrac{z}{3z+1}$

Put $z=i \Rightarrow f(z)=\dfrac{i}{3i+1}\times\dfrac{1-3i}{1-3i}=\dfrac{3+i}{10}$

$\therefore$ For $z=i$, $f(z)=\dfrac{3}{10}+\dfrac{1}{10}i$, i.e., $x>0, y>0$ or $u>0, v>0$

$\Rightarrow$ Either H^+ maps to H^+ or L^+

$\therefore$ options (b) and (d) are incorrect.

Further, put $z=-i \Rightarrow f(z)=\dfrac{-i}{1-3i}=\dfrac{-i}{1-3i}\times\dfrac{1+3i}{1+3i}=\dfrac{3-i}{10}$

$\therefore$ For $z=-i$, $f(z)=\dfrac{3}{10}+\dfrac{1}{10}i$, i.e., $x>0, y<0$ or $u>0, y<0$

$\Rightarrow$ Either H^- maps to H^- or L^+

$\therefore$ option (c) is incorrect.

$\therefore$ option (a) is correct.

Example 9. *Let U be an open subset of $\mathbb{C}$ containing $\{z\in \mathbb{C}:|z|\le 1\}$ and let $f:U\to\mathbb{C}$ be the map defined by $f(z)=e^{i\psi}\dfrac{z-a}{1-\bar{a}z}$ for $a\in D$, and $\psi\in[0,2\pi]$. Which of the following statements are true?*

(CSIR UGC NET DEC-2011)

(a) $|f(e^{i\theta})|=1$ for $0\le\theta\le 2\pi$

(b) f maps $\{z\in \mathbb{C}:|z|<1|\}$ onto itself

(c) f maps $\{z\in \mathbb{C}:|z|<1|\}$ into itself

(d) f is one one

Solution: We know that a bilinear transformation of the form $w=e^{i\lambda}\left(\dfrac{z-\alpha}{\bar{\alpha}z-1}\right)$, $|\alpha|<1$

(i) maps $|z|=1$ onto $|w|=1$

(ii) maps $|z|<1$ into $|w|<1$

As for $z=e^{i\theta}$, $|f(z)|=1$

$\therefore$ option (a) is correct

By point (ii), option (c) is correct and option (b) is incorrect.

Also, Bilinear transformation is always one-one.

$\therefore$ Option (d) is correct.

Example 10. *Let $f(z)=z+\dfrac{1}{z}$ for $z\in\mathbb{C}$ with $z\neq 0$. Which of the following are always true?*

(CSIR UGC NET DEC-2012)

(a) f is an analytic function on $\mathbb{C}\setminus\{0\}$.

(b) f is a conformal map on $\mathbb{C}\setminus\{0\}$.

(c) f maps the unit circle to a subset of the real axis.

(d) *The image of any circle in* $\mathbb{C}\setminus\{0\}$ *is again a circle.*

Solution: (a,c) $f(z)=z+\frac{1}{z}$ *for* $z\in\mathbb{C}$ *with* $z\neq 0$

Clearly singularity of $f(z)=z+\frac{1}{z}$ *is* $z=0$ *only*

$\Rightarrow$ *f is analytic function on* $\mathbb{C}\setminus\{0\}$

$\Rightarrow$ *option (a) is correct*

For option (b),

$f'(z)=1-\frac{1}{z^2}$ *exists for all* $z\neq 0$ *and* $f'(z)=0\Rightarrow 1-\frac{1}{z^2}=0\Rightarrow z^2-1=0$

$\Rightarrow z=\pm 1$ $\Rightarrow$ *f (z) is not conformal in* $\mathbb{C}\setminus\{0\}$

$\therefore$ *option (b) is incorrect.*

Further, $f(z)=z+\frac{1}{z}$

On unit circle $|z|=1$, $f(z)=z+\bar{z}=2\operatorname{Re} z$

$\Rightarrow$ $f(z)=z+\frac{1}{z}$ *maps the unit circle to a subset of the real axis*

$\Rightarrow$ *option (c) is correct and (d) is incorrect.*

Example11. *Let* $H=\{z=x+iy\in\mathbb{C}:y>0\}$ *be the upper half plane and* $D=\{z\in\mathbb{C}:|z|<1\}$ *be the open unit disc. Suppose that f is a Mobius transformation, which maps H conformally onto D. Suppose that* $f(2i)=0$. *Pick each correct statement from below.* **(CSIR UGC NET JUNE-2016)**

(a) *f has a simple pole at* $z=-2i$.

(b) *f satisfies* $f(i)\overline{f(-i)}=1$.

(c) *f has an essential singularity at* $z=-2i$.

(d) $|f(2+2i)|=\frac{1}{\sqrt{5}}$.

Solution: (a,b,d)

We know that bilinear transformation $g(z)=e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ *maps* $\operatorname{Im} z\geq 0$ *onto* $|w|\leq 1$ *if* $\operatorname{Im}\alpha>0$

If $w=f(z)$

Given that 'f' maps $\operatorname{Im} z>0$ *onto* $|w|<1$ *and* $f(2i)=0$ $\therefore$ $f(z)=e^{i\lambda}\frac{(z-2i)}{z+2i}$

Clearly f(z) has a simple pole at $z=-2i$

Further, $|f(2+2i)|=\left|\frac{2+2i-2i}{2+2i+2i}\right|=\frac{2}{\sqrt{20}}=\frac{1}{\sqrt{5}}$ $[\because |e^{i\lambda}|=1]$

Also, $f(i)=e^{i\lambda}\left(\frac{i-2i}{i+2i}\right)=\frac{-1}{3}e^{i\lambda}$

$$f(-i)=e^{-i\lambda}\left(\frac{-3i}{i}\right)=-3e^{-i\lambda}$$

$\therefore f(i).\overline{f(-i)}=1$

Hence, options (a), (b) and (d) are correct and option (c) is incorrect.

ASSIGNMENT - 6.1

NOTE: CHOOSE THE BEST OPTION

1. A mapping S(z) is called linear transformation if

(a) $S(z) = \dfrac{az}{cz}$ (b) $S(z) = az$ (c) $S(z) = \dfrac{az+b}{cz+d}$ (d) $S(z) = az + b$

2. A bilinear transformation $w = \dfrac{az+b}{cz+d}$ having only one fixed point is called

(a) loxodromic (b) elliptic (c) parabolic (d) hyperbolic

3. Critical points of the bilinear transformation $w = \dfrac{az+b}{cz+d}$ are

(a) $z = -d/c, z = 0$ (b) $z = -d/c, z = \infty$ (c) $z = 0, z = \infty$ (d) $z = -d/c, z = -b/a$

4. The Jordan's inequality is

(a) $\dfrac{2}{\pi} \le \dfrac{\sin\theta}{\theta} \le 1$ (b) $2\pi < \sin\theta < 0$ (c) $\dfrac{2}{\pi} < \dfrac{\theta}{\sin\theta} < 1$ (d) none of these

5. $\displaystyle\int_0^\infty \frac{dx}{x^6+1} =$

(a) $\pi/3$ (b) $2\pi/3$ (c) $4\pi/3$ (d) none of these

6. Let $f = u(x, y) + iv(x, y)$ and $g = v(x, y) + iu(x, y)$ be non-zero analytic functions on $|z|<1$. Then, it follows that

(a) $f' \equiv 0$ (b) f is conformal on $|z|<1$
(c) $f \equiv kg$ for some real k (d) f is one to one

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

7. If z_1, z_2, z_3, z_4 are distinct points and T is any Mobius transformation then the cross ratio (z_1, z_2, z_3, z_4) is not equal to

(a) (Tz_1, Tz_2, z_3, z_4) (b) (Tz_1, Tz_2, Tz_3, z_4) (c) (Tz_1, Tz_2, Tz_3, Tz_4) (d) (z_1, z_2, Tz_3, Tz_4)

8. If $z_1 \neq z_2 \neq z_3 \neq z_4$ lie in C_∞. The cross ratio (z_1, z_2, z_3, z_4) is a real number if z_1, z_2, z_3, z_4 does not lies on

(a) triangle (b) parabola (c) circle (d) hyperbola

9. The mobius transformation may take

(a) circles into line (b) circle into circle
(c) circle into square (d) straight line into straight line

10. *The points which coincide with their transformations are not*
(a) *fixed points* (b) *critical points* (c) *bilinear points* (d) *conformal points*

11. *Bilinear transformation which does not transforms the unit circle* $|z| \le 1$ *into the unit circle* $|w| \le 1$ *is*
(a) $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ (b) $w = e^{i\lambda}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right)$
(c) $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ (d) $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z+1}\right)$

12. *Let the transformation* $f(z)=u(x, y)+iv(x, y)$ *is conformal, then*
(a) $|f'(z)| \neq 0$ (b) $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 0$ (c) $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \neq 0$ (d) $f(z)$ *is not necessarily analytic*

13. *Transformation* $f(z)=\frac{1}{z}$ *maps*
(a) $|z|<1$ *onto* $|f(z)|>1$ (b) $|z|<1$ *onto* $|f(z)|<1$
(c) $|z|>1$ *onto* $|f(z)|>1$ (d) $|z|>1$ *onto* $|f(z)|<1$

14. *Transformation* $f(z)=\frac{1}{z}$
(a) *always maps a circle into a circle*
(b) *always maps a circle into straight line*
(c) *maps circle into straight line if circle in z-plane passes through origin*
(d) *maps straight line into circle*

15. *Consider any bilinear transformation* $w=\frac{az+b}{cz+d}(ad-bc\neq 0)$, *where* $\Delta=(d-a)^2+4bc$, *then w has*
(a) *only one finite fixed point, if* $c \neq 0$ *and* $\Delta \neq 0$
(b) *two finite fixed points, if* $c \neq 0$ *and* $\Delta \neq 0$
(c) *one finite and other infinite fixed point, if* $c \neq 0$ *and* $a \neq d$
(d) *only one infinite fixed point, if* $c=0$ *and* $a \neq d$

16. *For the transformation* $w=\frac{z}{2-z}$,
(a) *fixed points are 0 and 2* (b) *fixed points are 0 and 1*
(c) *transformation is parabolic* (d) *transformation is hyperbolic*

17. *Which of the following(s) is/are true?*
(a) *Every bilinear transformation can be expressed as the resultant of translation, dialation and rotation.*
(b) *Every bilinear transformation can be expressed as the resultant of an even number of inversions.*
(c) *The set of all bilinear transformations form an abelian group under the product of transformations.*
(d) *The set of all bilinear transformations form a non abelian group under the product of transformations.*

ASSIGNMENT - 6.2

NOTE: CHOOSE THE BEST OPTION

1. *The point u is the reflection of the point a on the line* $z\bar{b} + \bar{z}b = c$ *if*

   (a) $\bar{u}b+\bar{a}b = c$ (b) $\bar{u}b+a\bar{b} = c$

   (c) $\bar{u}b+a\bar{b} = 0$ (d) $u\bar{b}+a\bar{b} = c$

2. *There is only one value of z for which w = z in bilinear transformation* $w = \dfrac{az+b}{cz+d}$, *if*

   (a) $(a-d)^2 + 4bc = 0$ (b) $(a-d)^2 + 4bc \neq 0$

   (c) $(a-d)^2 = 4bc$ (d) $(a-d)^2 \neq 4bc$

3. *Bilinear transformation which maps the half plane* $Im(z) \geq 0$ *onto the circular disc* $|w| \leq 1$ *is*

   (a) $w = e^{i\lambda}\left(\dfrac{z-\alpha}{z-\bar{\alpha}}\right)$ (b) $w = e^{i\lambda}\left(\dfrac{z-\alpha}{z+\bar{\alpha}}\right)$

   (c) $w = e^{i\lambda}\left(\dfrac{z-\alpha}{\bar{\alpha}z-1}\right)$ (d) none of these

4. *Transformation* $w = \dfrac{az+b}{cz+d}$ *transforms the unit circle in the w-plane into straight line in the z-plane if*

   (a) b/a (b) $|a| = |c|$ (c) $bc + ad = 0$ (d) $bc - ad = 0$

5. *If the cross ratio* (z_1, ∞, z_2, z_3) *is real, then*

   (a) (z_1, z_2, z_3) *are collinear*

   (b) (z_1, z_2, z_3) *are concyclic*

   (c) z_1, z_2 *and* z_3 *are collinear when at least one of* z_1, z_2, or, z_3 *is real*

   (d) z_1, z_2 *and* z_3 *are concyclic when at least one of* z_1, z_2, or, z_3 *is real*

6. *Critical points of the transformation* $w = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ *are*

   (a) $-\infty, 0$ (b) $0, \infty$ (c) ± 1 (d) $\pm\infty$

7. *If AB be the arc* $\alpha \leq \theta \leq \beta$ *of the circle* $|z| = R$ *and* $\lim\limits_{z\to\infty} zf(z) = k$, *then*

   (a) $\lim\limits_{R\to\infty} \int\limits_{AB} f(z)dz = i(\beta-\alpha)k$ (b) $\lim\limits_{R\to 0} \int\limits_{AB} f(z)dz = i(\beta-\alpha)k$

   (c) $\lim\limits_{R\to\infty} \int\limits_{AB} f(z)dz = (\beta-\alpha)k$ (d) $\lim\limits_{R\to 0} \int\limits_{AB} f(z)dz = (\beta-\alpha)k$

8. A linear transformation which takes the triangle $\Delta(z) = (0, 1, i)$ in the z- plane into the triangle $\Delta(w) = (-1, -i, i)$ in the w-plane is
(a) $w = (1-i)z - 1$ (b) $w = (1+i)z - 1$ (c) $w = (1-i)z + 1$ (d) $w = (1+i)z + 1$

9. The magnification factor of the conformal mapping $w = \sqrt{2}e^{\pi i/4}z + (1-2i)$ is
(a) 1 (b) 2 (c) 3 (d) $\sqrt{2}$

10. Let $f(z) = (z - z_0)^{100}e^z$, then the mapping $w = f(z)$ magnifies the angle at the vertex z_0 by the factor K, where K is
(a) 1 (b) 100 (c) 101 (d) 0

11. By the transformation $w = ze^{i\pi/4}$, the line x=0 is transformed into the line
(a) v=-u (b) v=u (c) u+v=1 (d) v=0

12. Under the transformation $w = z + 1 - i$, the image of the line y=0 in the z-plane is
(a) v=1 (b) v=-1 (c) u=1 (d) u=0

13. Under the transformation $w = \frac{1}{z}$, the image of the line $y = \frac{1}{4}$ in z-plane is
(a) circle $u^2 + v^2 + 4v = 0$ (b) circle $u^2 + v^2 = 4$
(c) circle $u^2 + v^2 = 2$ (d) none of the above

14. The mapping $w = z^2 - 2z - 3$ is
(a) conformal within $|z| = 1$ (b) not conformal at z=1
(c) not conformal at z=-1 and z=3 (d) conformal everywhere

15. Transformation $w = z^{1/\alpha}$ maps
(a) half planes into circle (b) wedges or sectors into half planes
(c) half planes into conformal hyperbolas (d) none of the above

16. Which of the following transformation is not conformal in a unit disc centered at origin?
(a) $e^{\sin z}$ (b) $e^{\log z}$ (c) $e^{\sin z^2}$ (d) $e^{\tan z}$

17. The conjugate point of $1 + i$ with respect to the circle $|z - 1| = 2$ is
(a) $1 - i$ (b) $1 + 2i$ (c) $1 + 4i$ (d) $-1 - i$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

18. Let $w(z) = \frac{az+b}{cz+d}$ and $f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ be the bilinear transformations. Then which of the following may not be a bilinear transformation

(a) $f(z)\,w(z)$ (b) $f(w(z))$ (c) $f(z)+w(z)$ (d) $f(z)+\dfrac{1}{w(z)}$

19. *The non constant transformation* $w=\bar{z}$ *is*
(a) conformal (b) isogonal (c) not conformal (d) not isogonal

20. *Critical points of mapping* $w=e^{2z}-2iz+3$ *are*
(a) $2n\pi+\dfrac{\pi}{2}$ (b) $n\pi+\dfrac{\pi}{4}$ (c) $i\left(2n\pi+\dfrac{\pi}{2}\right)$ (d) $i\left(n\pi+\dfrac{\pi}{4}\right)$

21. *Which of the following statements is (are) false? Constant map has*
(a) infinite critical points
(b) finite critical point
(c) no critical points
(d) none of these

22. *Which of the following(s) is/are true?*
(a) Composite of two bilinear transformations is bilinear
(b) Every bilinear transformation (except identity map) has at most two fixed points
(c) Bilinear transformation can map straight line to a circle
(d) Bilinear transformation can map circle to a circle

23. *Which of the following(s) is/are correct?*
(a) The cross ratio is invariant under bilinear transformation
(b) The bilinear transformation, which maps -1,0,1 onto 0, $i,3i$ is $w=-\dfrac{3i(z+1)}{z-3}$
(c) Both (a) and (b) are true
(d) Neither (a) nor (b) is true

ASSIGNMENT - 6.3

NOTE: CHOOSE THE BEST OPTION

1. *The critical points of the transformation* $w^2 = (z-a)(z-b)$ *are*
   (a) $a+b$ (b) $(a+b)/2$ (c) $a-b$ (d) 0

2. *Let* $w = (1 - iz)/z$. *Then the unit circle* $|w| = 1$ *is mapped on the line*
   (a) $2x + 1 = 0$ (b) $2y + 1 = 0$ (c) $2x + 4 = 0$ (d) $y + 1 = 0$

3. *The image of* $|z + 1| = 1$ *under the mapping* $w = 1/z$ *is*
   (a) $2u-1=0$ (b) $2u+1=0$ (c) $u^2-1=0$ (d) $u=2$

4. *The image of the right half–plane* $x \geq 0$ *under the mapping* $w = (z-1)/(z+1)$ *is*
   (a) right half plane $u \geq 0$ (b) the upper half plane $v \geq 0$
   (c) the disc $|w| \leq 1$ (d) none of these

5. *If* $T_1(z)=\dfrac{z+2}{z+3}$ *and* $T_2(z) = \dfrac{z}{z+1}$, *then* $T_2^{-1}\,T_1(z)$ *is*
   (a) $z + 3$ (b) $z + 2$ (c) $z + 6$ (d) $z - 3$

6. *If* $w = T_1(z) = \dfrac{z+2}{z+3}$, *then* $T_1^{-1}(z)$ *is*
   (a) $\dfrac{2-3w}{w+1}$ (b) $\dfrac{2-3w}{w-1}$ (c) $\dfrac{1}{w+3}$ (d) none of these

7. *If* $T_1(z)=\dfrac{z+2}{z+3}$ *and* $T_2(z) = \dfrac{z}{z+1}$, *then* $T_2T_1(z)$ *is*
   (a) $z + 2$ (b) $\dfrac{2}{2z+5}$ (c) $\dfrac{z+2}{2z+5}$ (d) None of these

8. *If* z_1, z_2, z_3, z_4 *are distinct points in the order in which they are written, then number of distinct cross-ratios is*
   (a) $4!$ (b) 4^4 (c) 6 (d) 1

9. *The image of x=constant under the transformation* $w = \sin z$ *is*
   (a) a parabola (b) a hyperbola (c) a circle (d) an ellipse

10. *The bilinear transformation that maps the points* $z_1 = 2,\ z_2 = i,\ z_3 = -2$ *in the points* $w_1 = 1, w_2 = i, w_3 = -1$ *is*
    (a) $w=\dfrac{3z-2i}{iz-6}$ (b) $w=\dfrac{3z-2i}{iz+6}$ (c) $w=\dfrac{3z+2i}{iz-6}$ (d) $w=\dfrac{3z+2i}{iz+6}$

11. *Transformation* $w=\bar{z}$ *transforms*
(a) left half plane to right half plane
(b) lower half plane to left half plane
(c) upper half plane to right half plane
(d) upper half plane to lower half plane

12. *The bilinear transformation* $w=\dfrac{i(1-z)}{1+z}$ *maps*
(a) upper half plane into unit circle $|w|\le 1$
(b) the unit circle $|z|\le 1$ into the right half plane
(c) the unit circle $|z|<1$ into upper half plane
(d) the unit circle $|z|\le 1$ into lower half plane

13. *The analytic function which maps the annular region* $0\le\theta\le\frac{\pi}{4}$ *onto the upper half plane is*
(a) z^2
(b) $4z$
(c) z^4
(d) $2z$

14. *An annular domain in the complex plane is defined by* $0<\text{amp}(z)<\dfrac{\pi}{4}$. *The mapping which maps this region onto the left half plane is*
(a) $w=z^4$
(b) $w=iz^4$
(c) $w=-z^4$
(d) $w=-iz^4$

15. *If* $z=re^{i\theta}$, *then the image of* θ *=constant under the mapping* $w(z)=\mathrm{Re}^{i\phi}=iz^3$ *is*
(a) $\phi=3\theta$
(b) $\phi=3\theta+\dfrac{\pi}{2}$
(c) $\phi=3\theta-\dfrac{\pi}{2}$
(d) $\phi=\theta^3$

16. *The Mobius transformation maps the points* $1-2i, 2+i, 2+3i$, *respectively into* $2+2i, 1+3i, 4$ *is*
(a) $w=\dfrac{i(1-z)}{1+z}$
(b) $w=\dfrac{1-z}{1+z}$
(c) $w=\dfrac{(20+18i)-(32+12i)z}{(29+17i)-(11-3i)z}$
(d) $w=\dfrac{(20+18i)+(32+12i)z}{(29+17i)-(11+3i)z}$

17. *The angle through which a curve drawn from the point* $1+i$ *is rotated under the mapping* $f(z)=z^2$ *is*
(a) $\dfrac{\pi}{6}$
(b) $\dfrac{\pi}{4}$
(c) $\dfrac{\pi}{3}$
(d) $\dfrac{\pi}{2}$

18. *If* $f(z)=(z-2)^3\,g(z)$, *where* $g(z)$ *is analytic in* $|z-2|<r$ *and* $g(2)\neq 0$. *If angle at the vertex* $z=2$ *is* $\dfrac{\pi}{3}$, *then the angle at* $f(2)$ *is*
(a) $\dfrac{2\pi}{3}$
(b) π
(c) $\dfrac{\pi}{2}$
(d) $\dfrac{\pi}{3}$

19. *The bilinear transformation* $w=2z/(z-2)$ *maps* $\{z:|z-1|<1\}$ *onto*
(a) $\{w : \mathrm{Re}\, w<0\}$
(b) $\{w : \mathrm{Im}\, w>0\}$
(c) $\{w : \mathrm{Re}\, w>0\}$
(d) $\{w : |w+2|<1\}$

20. The function $w(z) = -\left(\frac{1}{z} + bz\right)$, $-1 < b < 1$, maps $|z| < 1$ onto

(a) a half plane
(b) exterior of the circle
(c) exterior of an ellipse
(d) interior of an ellipse

21. The transformation $w = e^{i\theta}\left(\frac{z-p}{\bar{p}z-1}\right)$, where p is a constant, maps $|z| < 1$ onto

(a) $|w| < 1$ if $|p| < 1$
(b) $|w| > 1$ if $|p| > 1$
(c) $|w| = 1$ if $|p| = 1$
(d) $|w| = 3$ if $p = 0$

22. The fixed points of $f(z) = \frac{2iz+5}{z-2i}$ are

(a) $1 \pm i$ (b) $1 \pm 2i$ (c) $2i \pm 1$ (d) $i \pm 1$

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

23. The rectangular region R bounded by $x = 0$, $y = 0$; $x = 2$, $y = 3$ in plane is mapped into the rectangle region R of w – plane through the transformation $w=\sqrt{2}e^{\frac{i\pi}{2}}z$, this transformation performs

(a) rotation (b) magnification (c) translation (d) inversion

24. Which of the following is correct if f(z) is conformal at z_0

(a) f(z)is analytic at z_0
(b) f(z) has local inverse at z_0
(c) z_0 is critical points of f(z)
(d) z is not critical point of f(z)

25. The fixed points of the transformation $w = (1 + z)/(1 - z)$ is

(a) -1 (b) 1 (c) i (d) -i

26. The image of the circle $|z| = 2$ under the map $w = 1/z$ is a

(a) circle
(b) straight line passing through the origin
(c) circle of radius of 1/2
(d) none of these

27. $\arg(z-i)=\frac{\pi}{3}$ represents

(a) a circle of radius $\pi/3$ and centre $(0, 1)$
(b) a circle of radius $\pi/3$ and centre $(1, 0)$
(c) an equation $y = \sqrt{3}x+1$
(d) a line

28. Function $w=\frac{z+i}{iz+1}$ does not maps

(a) $Im\ z \le 0$ onto $|w| \le 1$
(b) $Im\ z \ge 0$ onto $|w| \le 1$
(c) $Im\ z \le 0$ onto $|w| \ge 1$
(d) $Im\ z \ge 0$ onto $|w| \ge 1$

29. If w_1, w_2, w_3 and w_4 are the images of the four distinct points z_1, z_2, z_3 and z_4 in the z-plane under a bilinear transformation, then

(a) $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$
(b) $(w_1, w_2, w_3, w_4) = (z_1, z_3, z_4, z_2)$
(c) $\dfrac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$
(d) $(w_1, w_2, w_3, w_4) = (0,0,0,0)$

30. Which of the following is true for the transformation $w = \dfrac{iz+2}{4z+i}$?

(a) It maps the real axis in the z-plane into a circle in the w-plane
(b) It does not map the real axis in the z-plane into a circle in the w-plane
(c) It maps the real axis in the z-plane into a parabola in the w-plane
(d) It maps the real axis in the z-plane into a square in the w-plane

31. Which of the following statement(s) is/are true?

(a) The transformation $w = \dfrac{z+i}{z-i}$ transform $|w| \le 1$ into lower half of the plane $I(z) \le 0$
(b) The transformation $w = \dfrac{i+z}{i-z}$ transform $|w| \le 1$ into lower half of the plane $I(z) \le 0$
(c) The transformations $w = \dfrac{z+i}{z-i}$ and $w = \dfrac{i+z}{i-z}$ transform $|w| \le 1$ into the lower half plane $I(z) \le 0$
(d) For the transformation $w = \dfrac{z+i}{z-i}$ and $w = \dfrac{i+z}{i-z}$, $|w|^2 - 1 \le 0$

32. Consider the bilinear transformation from $\mathbb{C}_\infty$ to $\mathbb{C}$. $T(z) = \begin{cases} \infty, z = \infty, c = 0 \\ \dfrac{a}{c}, z = \infty, c \ne 0 \\ \infty, z = -\dfrac{d}{c}, c \ne 0 \\ \dfrac{(az+b)}{(cz+d)}, \text{ otherwise} \end{cases}$. Then,

(a) T is one-one
(b) T is onto
(c) Inverse of T, i.e., T^{-1} is a bilinear transformation
(d) none of the above

33. Consider the transformation w=sinz. Then,

(a) the rectangle $-\pi < x < \pi, c < y < d$ maps onto the elliptic ring cut along the negative y-axis

(b) *the straight lines x=constant map onto confocal hyperbola*
(c) *both (a) and (b) is true*
(d) *neither (a) nor (b) is true*

34. *Consider the transformation* $w=f(z)=\dfrac{e^z-i}{e^z+i}$. *Then,*

(a) *w is one – one conformal mapping of the horizontal strip* $0<y<\pi$ *onto the disc* $|w|<1$
(b) *x-axis is mapped onto the lower semi circle bounding the disc*
(c) *the line* $y=\pi$ *is mapped onto the upper semi circle*
(d) *none of the above*

35. *Which of the following(s) is/are correct?*

(a) *The bilinear transformation* $w=\dfrac{-iz+4i}{z}$ *maps the crescent – shaped region that lies inside the circle* $|z-2|<2$ *and outside the circle* $|z-1|>1$ *onto a horizontal strip in the upper half plane*
(b) *The transformation* $w=\dfrac{z-1}{z+1}$ *maps* $x>0$ *and the boundary* $x=0$ *on* $|w|\le 1$
(c) *Let* $w=\dfrac{i(1-z)}{1+z}$, *then* $Im(z)<1 \Rightarrow \mathrm{Im}(w)>0$
(d) *None of the above*

36. *The function f(z) that is holomorphic, within the unit circle which takes value* $\dfrac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}, |a|>1$, *is*

(a) $\dfrac{1}{a-z}$
(b) $\dfrac{1}{z-a}$
(c) $\dfrac{1}{z}$
(d) $\dfrac{a}{a-z}$

37. *Putting* $z=e^{i\theta}$ *such that C is the circle* $|z|=1$, *then the integral* $\int_0^{2\pi}\dfrac{\sin 3\theta}{5-3\cos\theta}d\theta$ *reduces to*

(a) $\oint_C \dfrac{(z^3-1)dz}{z^3(z-3)}$
(b) $\oint_C \dfrac{(z^6-1)dz}{z^3(3z-1)}$
(c) $\oint_C \dfrac{(z^6-1)dz}{z^3(z-3)(3z-1)}$
(d) *none of these*

38. *Which of the following is true for the transformation* $w=\dfrac{5-4z}{4z-2}$?

(a) *It transforms circle* $|z|=1$ *into circle of radius unity in w-plane*
(b) *It transforms circle* $|z|=1$ *into parabola in w-plane*
(c) *It transforms circle* $|z|=1$ *into square in w-plane*
(d) *It transforms circle* $|z|=1$ *into rectangle in w-plane*

ANSWERS TO EXERCISES

PRACTICE SET

Exercise 1: *(c)*
Exercise 2: *(a)*
Exercise 3: *(a)*
Exercise 4: *(b)*
Exercise 5: *(a,b,d)*

ANSWERS TO ASSIGNMENTS

ASSIGNMENT – 6.1

1. (d)	2. (c)	3. (b)	4. (a)	5. (a)	6. (a)	
7. (a,b,d)	8. (a,b,d)	9. (a,b,d)	10. (b,c,d)	11. (a,b,d)	12. (a,c)	13. (a,d)
14. (a,c)	15. (b)	16. (b,d)	17. (a,b,d)			

ASSIGNMENT – 6.2

1. (b)	2. (a)	3. (a)	4. (b)	5. (a)	6. (c)	7. (a)
8. (a)	9. (d)	10. (b)	11. (a)	12. (b)	13. (a)	14. (b)
15. (b)	16. (c)	17. (c)				
18. (a,c,d)	19. (b,c)	20. (d)	21. (a,b,d)	22. (a,b,c,d)	23. (a,b,c)	

ASSIGNMENT – 6.3

1. (b)	2. (b)	3. (b)	4. (c)	5. (b)	6. (b)	7. (c)
8. (c)	9. (b)	10. (d)	11. (d)	12. (c)	13. (c)	14. (b)
15. (b)	16. (c)	17. (b)	18. (b)	19. (a)	20. (b)	21. (a)
22. (c)						
23. (a,b)	24. (a,b,d)	25. (c,d)	26. (a,c)	27. (c,d)	28. (b,c)	29. (a,c)
30. (a)	31. (a,b,c,d)	32. (a,b,c)	33. (a,b,c)	34. (a,b,c)	35. (a,b,c)	36. (a)
37. (c)	38. (a)					

ABOUT THE AUTHOR

Dr A.P. Singh is Ph.D. in Mathematics from Panjab University. He has been a Gold Medalist in his Graduation as well as Post-Graduation and a recipient of National Board of Higher Mathematics (NBHM) scholarship besides being a sportsman and a chess lover. He cleared his CSIR UGC JRF in December 2000 while he was pursuing his Masters Degree. During his Ph.D., he has been conferred with the accolades like Indian Young Scientist Award, Prof M. Venkataraman Best Paper Presentation Award. After completing studies, he earned name and fame for his sustained efforts in imparting quality education to his students many of whom are now pursuing research in IITs, IISERs, Panjab University, Delhi University and other reputed institutions. For the last 15 years, Dr A.P. Singh is committed to promoting higher Mathematics by his unique way of teaching, guiding and motivating students to crack the national level tests like NET, GATE, TIFR, IIT-JAM, etc., so that they may pursue their career in Higher Mathematics.

Dr A.P. Singh is always willing to take up students' problems on 9876788051 and on facebook group "Higher Maths discussion by Dr A.P. Singh".

INFOSTUDY PUBLICATIONS

BE INFORMED BE LEARNED

www.infostudypublications.com

ISBN : 978-81-933363-5-9

INR ₹ : 675/-